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COMPUTER ANALYSIS OF RING-STIFFENED SHELLS OF REVOLUTION

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16. Abstract

This report presents the equations and method of solution for a series of five compatible computer programs for structural analysis of axisymmetric shell structures. User manuals andother program documentation for these programs are presented in a separate companion report. These programs, designated as the SRA programs, apply to a common structural model but analyze different modes of structural response. They are:

(1) Linear asymmetric static response (SRA 100)
(2) Buckling of linearized asymmetric equilibrium states (SRA 101)
(3) Nonlinear axisymmetric static response (SRA 200)
(4) Buckling of nonlinear axisymmetric equilibrium states (SRA 201)
(5) Vibrations about nonlinear axisymmetric equilibrium states (SRA 300)

The theory of a sixth related program, for the imperfection sensitivity analysis of buckling modes of nonlinear axisymmetric equilibrium states, has been presented in a previous NASA report.

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SUMMARY

This report presents the equations and method of solution for a series of five compatible computer programs for structural analysis of axisymmetric shell structures. User manuals and other program documentation for these programs are presented in a separate companion report. These programs, designated as the SRA programs, apply to a common structural model but analyze different modes of structural response. They are:

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The theory of a sixth related program, for the imperfection sensitivity analysis of buckling modes of nonlinear axisymmetric equilibrium states, has been presented in a previous NASA report.

The structural model treated is a branched shell of revolution with an arbitrary arrangement of a large number of open branches but with at most one closed branch. The shell wall is assumed to be of orthotropic material with principal axes of orthotropy in meridional and circumferential directions. Geometric properties of the structure may vary only in the meridional direction; material properties of the shell wall may vary in the thickness direction as well as the meridional direction. Also treated are:

(1) discrete irotropic ring attachments,
(2) isotropic stringers, whose stiffness is circumferentially distributed, and
(3) an idealized elastic foundation attached to the shell wall.

INTRODUCTION

During the past decade an almost bewildering variety of computer programs has been developed for the analysis of shell structures (ref. 1). When one narrows the field to those designed for elastic shells of revolution, he is still confronted with the names of at least forty authors in this country alone who have been active in developing programs of overlapping capabilities (refs. 1 and 2). At the time reference 2 was written, however, there were known to be only four major systems which cover the most common problems of stress, buckling, and vibration of elastic shells of revolution. In addition to the SRA programs of this report, these include two finite-difference programs, BOSOR (ref. 3) and SALORS (refs. 4 and 5), and Kalnins' forward integration programs (ref. 6).

The SRA programs employ the Zarghamee version of the forward integration method (ref. 7) for the solution of the basic linear boundary value problem. This method requires the calculation of only four complementary solutions, as opposed to the usual eight, over open branches. The main features of the present system of programs which have not been generally available in the other systems are:

(1) buckling analysis under general asymmetric loads,
(2) imperfection sensitivity analysis, and
(3) branched shell capability (see fig. 1)

User documentation for the present system of six programs is presented in a companion report (ref. 8). As these programs have been developed over a period of time, the theory underlying some of them has already been published in the open literature (refs. 9-12). The theory of the nonlinear axisymmetric response program and the buckling program for general asymmetric equilibrium states, which is a new program, have not been previously presented. The purpose of this report is to bring together the underlying equations and (improved) method of solution for each of these programs except the imperfection sensitivity program, the theory of which has been presented in a previous NASA report (ref. 13).

SYMBOLS

A	ring or stringer cross-sectional area
a	ring centroidal radius
$C_1^{(i)}, C_2^{(i)}, C_{12}^{(i)}$	shell wall normal stiffnesses, eqs. (A-3)
E	ring or stringer elastic modulus
E_1, E_2, E_{12}	orthotropic elastic moduli
e_x, e_y	ring centroidal eccentricities relative to corresponding boundary point on shell reference surface
e_z	normal eccentricity of stringer centroid relative to shell reference surface
e_1, e_2, e_{12}	linearized shell stretching strains
F_x, F_y, F_ϕ	effective ring force loads per unit of circumferential length
F_1, F_2, F_3, F_4	equivalent shell forces, eqs (78)
FL_1, FL_2, FL_3, FL_4	equivalent ring forces, eqs. (79)
GJ	ring or stringer torsional stiffness
$G^{(i)}$	shell wall shear stiffnesses, eqs. (A-3)
I	stringer section moment of inertia about circumferential centroidal axis
I_x, I_y, I_{xy}	ring section moments of inertia
K	structural stiffness
k_1, k_2, k_3	elastic foundation moduli
L_1, L_2	effective shell moment loads per unit of area
M_x, M_y, M_ϕ	ring stress couples
M_1, M_2, M_{12}	shell stress couples

$m^{(i)}$	mass coefficients for shell inertial loads, eq. (109)
N	number of stringers
N_x, N_y, N_ϕ	effective ring moment loads per unit of circumferential length
n	circumferential harmonic number
P, Q, S	effective shell forces per unit of circumferential length in axial, radial, and circumferential directions, respectively
p	local pressure for live pressure field at unit λ
R_1, R_2	meridional and circumferential radii of curvature
r	small circle radius
s, ϕ, z	meridional, circumferential, and normal coordinates, respectively, of shell reference surface
T_1, T_2, T_{12}	shell stress resultants
T_ϕ	ring hoop force
U	ring potential energy
u, v, w	shell displacements in meridional, circumferential, and normal directions, respectively
u_x, u_y, u_ϕ	ring centroidal displacements
w_x, w_y, w_ϕ	ring rotations
X_1, X_2, X_3	effective shell force loads per unit of area
x, y	axial and radial coordinates, respectively
$\hat{z}$	normal distance of reference surface from shell inner surface
$\varepsilon_1, \varepsilon_2, \varepsilon_{12}$	shell stretching strains
ε_ϕ	ring hoop strain

$\Theta_1^{(i)},\Theta_2^{(i)},\Theta_{12}^{(i)}$	effective thermal loads (i = 0 or 1), eqs. (9)
$\theta_1,\theta_2,\theta_{12}$	effective free thermal strains
θ_{st}	stringer free thermal strain
θ_ϕ	ring effective free thermal strain
κ_x,κ_y,τ	ring bending strains
$\kappa_1,\kappa_2,\kappa_{12}$	shell bending strains
λ	load factor (for proportional loading)
λ_0	load factor for nonlinear prebuckling state
λ^*	limit load
λ_{ij}	orthotropic shell wall normal stiffness coefficients
μ	$\lambda - \lambda_0$
μ_i	eigenvalues
μ_{ij}	orthotropic shell wall shear stiffness coefficients
ν_1,ν_2	Poisson contraction ratios with meridional or circumferential stress acting, respectively
ξ,η,v	shell displacements in axial, radial, and circumferential directions, respectively
ρ	mass density
$\sigma_s,\sigma_\phi,\sigma_z$	three-dimensional normal stress components
$\sigma_{s\phi},\sigma_{sz},\sigma_{\phi z}$	three-dimensional shear stress components
χ,ψ,θ	shell rotations about circumferential, meridional, and normal directions, respectively
ω	frequency of harmonic vibrations

Vectors:

$\underset{\sim}{F}$	$\underset{\sim}{Y}'$

$\underset{\sim}{Y}$	eight (or six) element column vector of dependent variables $\{P,Q,S,M_1,\xi,\eta,v,\chi\}$
$\underset{\sim}{Y}_p$	particular solution
$\underset{\sim}{Y}_c^{(k)}$	complementary solutions

4x4 (or 3x3) Matrices:

[B],[D]	boundary condition matrices, eq. (25)
$[\tilde{D}]$	effective [D] for interior boundaries, eq. (48a)
$[\hat{D}]$	additional effective [D] for closed branch boundaries, eq. (55)
[e]	ring eccentricity matrix, eq. (34a)
[k]	ring stiffness matrix, eq. (30a)
$[p],[\hat{p}]$	matrices relating {c}, {d} of first subinterval to that of final subinterval of a closed branch, eqs. (59)
[S]	scaling matrix for supplemental conditions, eqs. (42)
[U],[W]	force and displacement submatrices of complementary solutions, eq. (36a)
[V],[Z]	additional force and displacement submatrices of complementary solutions required on closed branch
$[\kappa]$	ring prestress matrix, eq. (103)
$[\mu]$	ring mass matrix, eq. (112)

4x1 (or 3x1) Matrices:

{c}	superposition constants, eqs. (37)
{d}	additional superposition constants for a closed branch, eqs.(52)
{G},{J}	force and displacement submatrices of particular solution vector, eq. (36b)

$\{L\}$	effective boundary loads, eq. (25)
$\{\tilde{L}\}$	effective $\{L\}$ for interior boundaries, eq. (48b)
$\{\ell_e\}$	nonhomogeneous ring matrix due to ring eccentricity and thermal loads, eq. (34b)
$\{\ell_f\}$	nonhomogeneous ring matrix due to mechanical loads, eq. (30c)
$\{\ell_t\}$	nonhomogeneous ring matrix due to thermal loads, eq. (30d)
$\{\ell_f^{(e)}\}$	$\{\ell_f\}$ associated with externally applied loads
$\{q\}$	nonhomogeneous matrix relating $\{c\}$ of first subinterval to that of final subinterval of a closed branch, eq. (59a)
$\{u\}$	ring displacements $\{u_x, u_y, u_\phi, w_\phi\}$
$\{y\}$	shell forces $\{P, Q, S, M_1\}$
$\{z\}$	shell displacements $\{\xi, \eta, v, \chi\}$

Generalized field variables and operators:

$H(\varepsilon)$	linear operator relating stresses and strains
$L_1(u)$	linear operator representing linear part of the strain-displacement relations
$L_2(u)$	quadratic operator representing the nonlinear part of the strain-displacement relations
$L_{11}(u,v)$	bilinear operator defined by the identity $L_2(u + v) = L_2(u) + 2L_{11}(u,v) + L_2(v)$
$q_1(u)$	linear operator representing conservative live loads
u	displacement
ε	strain
σ	stress
$u_i, \varepsilon_i, \sigma_i$	eigenfunctions

Subscripts:

0	prebuckling state variable
1,2,3	meridional, circumferential, and normal components, respectively (same as s,ϕ,z)
$(\ )_{(k)}$	estimate after k iterations

Superscripts:

$(\ )^{(a)}$	antisymmetric component
$(\ )^{(k)}$	$\partial^k(\ )/\partial\lambda^k$
$(\ )^{(s)}$	symmetric component
$(\ )^T$	transpose
$(\bar{\ })$	load or linear response variable at unit λ
$(\ )'$	$\partial(\ )/\partial s$
$(\ )^{\cdot}$	$\partial(\ )/\partial\phi$
$(\dot{\ })$	$\partial(\ )/\partial r$

Matrix subscripts:

0	evaluated at the initial point of a subinterval
1	evaluated at the final point of a subinterval

GOVERNING EQUATIONS

Mathematically speaking, elastic response problems of shell structures are boundary-value problems in differential equations. In general, to formulate such problems, it is necessary to start with a geometrically nonlinear shell theory, i.e., one valid for rotations of moderate size.* Also, an analogous theory for elastic rings must be available to formulate boundary conditions associated with ring attachments. As a preliminary to the formulation of specific types of response problems solved by the SRA programs, suitable nonlinear theories of shells of revolution and rings are presented in this section.

*In this approximation, both the strains and rotations are small compared to unity, but the rotations may considerably exceed the strains.

Shell Equations

Nonlinear shell theories have been developed by Sanders (ref. 14) and others. However, for the purpose of numerical analysis of shells of revolution, it has been shown that Novozhilov's shell equations (ref. 15) have the advantage that, by the proper choice of dependent variables, explicit reference to the meridional radius of curvature can be eliminated (ref. 9).

In reference 11, Novozhilov's equations have been generalized, through the principal of virtual work, to include the nonlinear case of moderate rotations. For numerical analysis, it is convenient to transform the equilibrium and kinematic equations into a set of eight differential equations in eight basic force and displacement shell variables referred to fixed coordinate directions. Four of these variables are the effective shell forces in axial, radial, and circumferential directions, denoted as P, Q, and S respectively, and the meridional bending moment M_1. These components act on normal sections tangent to small circles of the shell reference surface (fig. 1) and are all measured per unit of circumferential length along the small circle. The remaining four variables are the analogous reference surface displacements, denoted as ξ, η, and v, and rotation χ. These variables, as well as the notation used for other shell variables, are shown in figure 2. As shown, s,ϕ reference surface coordinates are used where s measures meridional arc distance from a reference small circle and ϕ measures circumferential angle from a reference meridian. The normal distance z measured from the reference surface completes the three-dimensional triad of directions.

The transformation of the equations is accomplished with the use of the Gauss-Codazzi surface compatibility relations. Employing the prime and dot to denote partial derivatives with respect to s and ϕ, respectively, the resulting equilibrium equations are

$$
\begin{aligned}
&(rP)' + (r/R_2)\dot{S} - (2/r)\dot{M}_{12} - (r'/r)\ddot{M}_2 - (r/R_2)[(T_1 + T_2)\theta]^{\boldsymbol{\cdot}} \\
&\qquad + r'(T_2\psi + T_{12}\chi)^{\boldsymbol{\cdot}} + r[(r/R_2)X_1 - r'X_3] + r'\dot{L}_1 = 0 \\
&(rQ)' + r'\dot{S} - T_2 + \ddot{M}_2/R_2 - r'[(T_1 + T_2)\theta]^{\boldsymbol{\cdot}} - (r/R_2)(T_2\psi + T_{12}\chi)^{\boldsymbol{\cdot}} \\
&\qquad + r[r'X_1 + (r/R_2)X_3] - (r/R_2)\dot{L}_1 = 0 \\
&(rS)' + r'S + \dot{T}_2 + \dot{M}_2/R_2 - r'(T_1 + T_2)\theta - (r/R_2)(T_2\psi + T_{12}\chi) \\
&\qquad + rX_2 - (r/R_2)L_1 = 0 \\
&(rM_1)' + r[r'P - (r/R_2)Q] - r'M_2 + 2\dot{M}_{12} - r(T_1\chi + T_{12}\psi) + rL_2 = 0
\end{aligned}
\qquad (1)
$$

where the surface force (X_1,X_2,X_3) and moment (L_1,L_2) components are referred to undeformed coordinate directions (fig. 2).

The nonlinear terms in equations (1) can be conveniently thought of as the following additional load terms applied to the linearized equations.

$$X_1 = -[(T_1 + T_2)\theta]^{\bullet}/r \tag{2a}$$

$$X_2 = -(r'/r)(T_1 + T_2)\theta \tag{2b}$$

$$X_3 = 0 \tag{2c}$$

$$L_1 = T_2\psi + T_{12}\chi \tag{2d}$$

$$L_2 = -(T_1\chi + T_{12}\psi) \tag{2e}$$

Additional effective surface loads dependent on the shell deformation arise in the cases of an elastic foundation attached to the shell wall and loading by a live normal pressure field. An orthotropic elastic foundation is considered under the assumption that it produces reactions per unit of surface area in meridional, circumferential, and normal directions which are proportional to the corresponding shell displacements at the surface to which it is attached. It is assumed that the attachment surface is the shell inner surface (i.e., the surface of inward pointing positive z-direction). In terms of the displacement of the reference surface the foundation loads are

$$X_1 = -k_1(u - \hat{z}\chi) \tag{3a}$$

$$X_2 = -k_2(v - \hat{z}\psi) \tag{3b}$$

$$X_3 = -k_3 w \tag{3c}$$

$$L_1 = \hat{z}X_2 \tag{3d}$$

$$L_2 = -\hat{z}X_1 \tag{3e}$$

where k_1, k_2, and k_3 are foundation moduli and $\hat{z}$ is the normal distance of the reference surface from the inner surface. The effective loads of a live pressure field $\lambda p(s,\phi,z)$, assumed to act at the reference surface, are

$$X_1 = \lambda p\chi \tag{4a}$$

$$X_2 = \lambda p\psi \tag{4b}$$

$$X_3 = \lambda[p(e_1 + e_2) + u\partial p/\partial s + v\partial p/\partial\phi + w\partial p/\partial z] \tag{4c}$$

$$L_1 = 0 \tag{4d}$$

$$L_2 = 0 \tag{4e}$$

where e_1 and e_2 are the linearized stretching strains in meridional and circumferential directions, respectively.

Equations (2), (3), and (4) isolate all terms of equations (1) other than standard terms of a linear shell statics problem.

The four basic kinematic equations may be written in the form

$$
\begin{aligned}
\xi' &= r'\chi + (r/R_2)e_1 \\
\eta' &= -(r/R_2)\chi + r'e_1 \\
v' &= -\dot{\xi}/R_2 - (r'/r)(\dot{\eta} - v) + e_{12} \\
\chi' &= \kappa_1
\end{aligned}
\qquad (5)
$$

where e_{12} is the linearized shearing strain and κ_1 is the meridional bending strain.

Equations (1) and (5) are eight partial differential equations in the eight response variables $P,Q,S,M_1,\xi,\eta,v,\chi$. Supplemental equations are necessary to express the excess variables of these equations in terms of the eight basic variables. The nonlinear strain-rotation equations and the partially inverted constitutive equations provide some of the supplemental equations. These are

$$
\begin{aligned}
e_1 &= \varepsilon_1 - (1/2)(\chi^2 + \theta^2) \\
e_2 &= \varepsilon_2 - (1/2)(\psi^2 + \theta^2) \\
e_{12} &= \varepsilon_{12} - \chi\psi
\end{aligned}
\qquad (6)
$$

and

$$T_2 = \lambda_{11}\varepsilon_2 + \lambda_{12}\kappa_2 + \lambda_{13}\tilde{T}_1 + \lambda_{14}\tilde{M}_1 - \Theta_2^{(0)} \qquad (7a)$$

$$M_2 = \lambda_{21}\varepsilon_2 + \lambda_{22}\kappa_2 + \lambda_{23}\tilde{T}_1 + \lambda_{24}\tilde{M}_1 - \Theta_2^{(1)} \qquad (7b)$$

$$\varepsilon_1 = \lambda_{31}\varepsilon_2 + \lambda_{32}\kappa_2 + \lambda_{33}\tilde{T}_1 + \lambda_{34}\tilde{M}_1 \qquad (7c)$$

$$\kappa_1 = \lambda_{44}\varepsilon_2 + \lambda_{42}\kappa_2 + \lambda_{43}\tilde{T}_1 + \lambda_{44}\tilde{M}_1 \qquad (7d)$$

$$2M_{12} = \mu_{11}\tilde{\kappa}_{12} + \mu_{12}\tilde{S} - 2\Theta_{12}^{(1)} \qquad (7e)$$

$$\varepsilon_{12} = \mu_{21}\tilde{\kappa}_{12} + \mu_{22}\tilde{S} \qquad (7f)$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_{12}$ and $\kappa_1,\kappa_2,\kappa_{12}$ are reference surface stretching and bending strains, respectively, and

$$\tilde{T}_1 = T_1 + \Theta_1^{(0)} \tag{8a}$$

$$\tilde{M}_1 = M_1 + \Theta_1^{(1)} \tag{8b}$$

$$\tilde{\kappa}_{12} = \kappa_{12} - \varepsilon_{12}/R_2 \tag{8c}$$

$$\tilde{S} = S - (1/2)(T_1 + T_2)\theta + \Theta_{12}^{(0)} + 2\Theta_{12}^{(1)}/R_2 \tag{8d}$$

The thermal loads $\Theta_1^{(m)}$, $\Theta_2^{(m)}$, and $\Theta_{12}^{(m)}$, for m = 0 or 1, are given in terms of the free thermal strains θ_1, θ_2, and θ_{12} by*

$$\begin{aligned} \Theta_1^{(m)} &= \int [E_1/(1 - \nu_1\nu_2)](\theta_1 + \nu_2\theta_2)z^m dz + \Theta_{st}^{(m)} \\ \Theta_2^{(m)} &= \int [E_2/(1 - \nu_1\nu_2)](\theta_2 + \nu_1\theta_1)z^m dz \\ \Theta_{12}^{(m)} &= \int E_{12}\theta_{12}z^m dz \end{aligned} \tag{9}$$

where E_1, E_2, E_{12}, ν_1, and ν_2 are orthotropic shell wall elastic moduli, the integrals are through the shell wall thickness, and $\Theta_{st}^{(m)}$ are given in terms of the stringer free thermal strain θ_{st} by

$$\begin{aligned} \Theta_{st}^{(0)} &= (NEA/2\pi r)\theta_{st} \\ \Theta_{st}^{(1)} &= e_z \Theta_{st}^{(0)} \end{aligned} \tag{9a}$$

Here NEA is the total stringer stretching stiffness and e_z is the normal eccentricity of stringer section centroids relative to the shell reference surface. The stiffness coefficients λ_{ij} and μ_{ij} in the constitutive equations (7) are defined in Appendix A.

In addition to equations (6) and (7), the following equations complete the supplemental equations.

$$T_{12} = S - 2M_{12}/R_2 - (1/2)(T_1 + T_2)\theta \tag{10}$$

and

$$\begin{aligned} T_1 &= (r/R_2)P + r'Q \\ e_2 &= (\eta + v^{\cdot})/r \\ \kappa_2 &= [r'(\chi + \xi^{\cdot\cdot}/r) + (v^{\cdot} - \eta^{\cdot\cdot})/R_2]/r \\ \kappa_{12} - e_{12}/R_2 &= (\chi - \xi/r)^{\cdot}/r \\ \psi &= (v - \eta^{\cdot})/R_2 + (r'/r)\xi^{\cdot} \\ \theta &= (r'/r)(v - \eta^{\cdot}) - \xi^{\cdot}/R_2 \\ u &= (r/R_2)\xi + r'\eta \\ w &= -r'\xi + (r/R_2)\eta \end{aligned} \tag{11}$$

---

*Although the shearing free thermal strain θ_{12} is zero for an orthotropic material, it will be seen to be convenient to include it in the formulation.

It may be noted that the only nonlinear terms appearing in equations (5) through (11) are those in equations (6),(8d), and (10). Just as the nonlinear terms in the equilibrium equations (1) may be viewed as additional mechanical loads applied to the linearized equations, the nonlinear terms in equations (6) and (8d) may be viewed as additional thermal loads applied to linearized versions of these equations. Noting that the three-dimensional strains $\varepsilon_1, \varepsilon_2, \varepsilon_{12}$ appear in the stress-strain relations as $\varepsilon_1 - \theta_1$, $\varepsilon_2 - \theta_1$, and $\varepsilon_{12} - \theta_{12}$, it follows that the nonlinear terms of equations (6) are equivalent to the following additional free thermal strains applied to the linearized equations,

$$\begin{aligned} \theta_1 &= -(1/2)(\chi^2 + \theta^2) \\ \theta_2 &= -(1/2)(\psi^2 + \theta^2) \\ \theta_{12} &= -\chi\psi \end{aligned} \tag{12a}$$

which do not vary through the shell thickness. The corresponding thermal loads are obtained by substituting equations (12a) into equations (9) to give

$$\begin{aligned} \Theta_1{}^{(m)} &= -(1/2)[C_s{}^{(m)}(\chi^2 + \theta^2) + C_{12}{}^{(m)}(\psi^2 + \theta^2)] \\ \Theta_2{}^{(m)} &= -(1/2)[C_2{}^{(m)}(\psi^2 + \theta^2) + C_{12}{}^{(m)}(\chi^2 + \theta^2)] \\ \Theta_{12}{}^{(m)} &= -G_s{}^{(m)}\chi\psi \end{aligned} \tag{12b}$$

where the stiffness coefficients $C_s{}^{(m)}$, $C_{12}{}^{(m)}$, $C_2{}^{(m)}$, and $G_s{}^{(m)}$ are defined by equations (A-3) of Appendix A. The nonlinear term in equation (8d) is evidently equivalent to the additional thermal load

$$\Theta_{12}{}^{(0)} = -(1/2)(T_1 + T_2)\theta \tag{13}$$

Finally, to identify the nonlinear term in equation (10) as an additional load, note that the only places where T_{12} is required are in equations (2d) and (2e). It therefore follows that this nonlinear term is equivalent to the additional loads*

$$\begin{aligned} L_1 &= -(1/2)(T_1 + T_2)\chi\theta \\ L_2 &= (1/2)(T_1 + T_2)\psi\theta \end{aligned} \tag{14}$$

Thus, the general nonlinear field equations may be viewed as a standard set of linearized equations with the additional load terms given by equations (2), (3), (4), (12b), (13), and (14).

---

*Since these loads are smaller then similar terms in equations (2d) and (2e) by a factor of the rotation θ, it is consistent with the moderate rotation theory to neglect the nonlinearity in equation (10).

Ring Equations

When ring stiffeners are attached to the shell, boundary conditions for the shell equations must be generated which represent the ring behavior. Ideally, the ring reactions enter the shell at a single meridional station, the ring boundary, on the shell reference surface at which the shell displacements are continuous and related to the shell force jumps in accordance with the governing ring equations. A set of suitable ring equations are derived in this section.

As are the shell equations of the previous section, the ring equations are based on moderate rotations and are derived through a principal of virtual work. These equations are based on the following assumptions.

(1) All geometrical and mechanical properties of the ring are axisymmetric.
(2) The ring material is homogeneous and isotropic.
(3) The effects of nonuniform warping of ring sections, transverse shear strains, and shear center eccentricity relative to the section centroid are neglected.

The origin of ring cross-sectional x,y-axes is supposed to be at the centroid of the ring section, i.e., $\int x dA = \int y dA = 0$ where A is the section area. With respect to right-handed shell coordinates s,ϕ,z, x is chosen positive in the axial direction acute to the positive (or negative) s-direction if the positive z-direction points away from (or towards) the axis of revolution, and y is chosen positive in the radial direction pointing away from the axis of revolution (see fig. 3).

For a one-dimensional theory of rings the centroidal hoop strain ε_ϕ is the only stretching strain of consequence. The strain-displacement relations are

$$\begin{aligned}
\varepsilon_\phi &= (\dot{u}_\phi + u_y)/a + (1/2)(w_x{}^2 + w_y{}^2) \\
\kappa_x &= (\ddot{u}_y - \dot{u}_\phi)/a^2 \\
\kappa_y &= -(\ddot{u}_x/a + w_\phi)/a \\
\tau &= (\dot{w}_\phi - \dot{u}_x/a)/a
\end{aligned} \qquad (15)$$

where κ_x and κ_y are the bending strains of the centroidal axis, in and out of the plane of this axis respectively, and τ is the twist per unit of circumferential length. Neglecting transverse shear strains, the rotations w_x, w_y may be written in terms of displacements as

$$
\begin{aligned}
w_x &= (\dot{u}_y - u_\phi)/a \\
w_y &= -\dot{u}_x/a
\end{aligned}
\tag{16}
$$

Integrating by parts the following expression for the virtual change in potential energy

$$
\begin{aligned}
\delta U = &\int_0^{2\pi} (T_\phi \delta\varepsilon_\phi + M_x \delta\kappa_x + M_y \delta\kappa_y + M_\phi \delta\tau) a\, d\phi \\
&- \int_0^{2\pi} (F_x \delta u_x + F_y \delta u_y + F_\phi \delta u_\phi + N_x \delta w_x + N_y \delta w_y + N_\phi \delta w_\phi) a\, d\phi
\end{aligned}
\tag{17}
$$

and applying the principle of virtual work, $\delta U = 0$, yields the following equations expressing equilibrium of forces and moments in the undeformed coordinate directions (fig. 3).

$$
\begin{aligned}
(\dot{M}_y/a - M_\phi/a + N_y - w_y T_\phi)^{\bullet} + aF_x &= 0 \\
(-\dot{M}_x/a - N_x + w_x T_\phi)^{\bullet} - T_\phi + aF_y &= 0 \\
\dot{T}_\phi - \dot{M}_x/a - N_x + w_x T_\phi + aF_\phi &= 0 \\
\dot{M}_\phi + M_y + aN_\phi &= 0
\end{aligned}
\tag{18}
$$

For a one-dimensional ring theory, the constitutive equations are unchanged from those for a straight elastic bar. Neglecting the effect of nonuniform torsion, for a homogeneous, isotropic bar these are (ref. 16)

$$
\begin{aligned}
T_\phi &= EA(\varepsilon_\phi - \theta_\phi) \\
M_x &= EI_x \kappa_x - EI_{xy} \kappa_y \\
M_y &= -EI_{xy} \kappa_x + EI_y \kappa_y \\
M_\phi &= GJ\tau
\end{aligned}
\tag{19}
$$

where the ring free thermal strain θ_ϕ is assumed to be uniform over each cross section.

In analogy with the nonlinear shell equations, the nonlinear terms in equations (15) and (18) may be viewed as effective additional moments and free thermal strain applied to the linearized ring equations.

These additional loads are

$$N_x = -w_x T_\phi \tag{20a}$$

$$N_y = -w_y T_\phi \tag{20b}$$

$$EA\theta_\phi = -(1/2)EA(w_x{}^2 + w_y{}^2) \tag{20c}$$

To reduce the ring equations to a more useful form, equations (15), (16), and (19) are substituted into equations (18) to eliminate all response variables except u_x, u_y, u_ϕ, and w_ϕ. With the understanding that the nonlinear terms are represented in the load terms according to equations (20), the resulting equations are

$$\begin{aligned}
&(EI_y \ddddot{u}_x - GJ\ddot{u}_x) + EI_{xy}(\ddddot{u}_y - \dddot{u}_\phi) + a(EI_y + GJ)\ddot{w}_\phi = a^3(aF_x + \dot{N}_y) \\
&EI_{xy}(\ddddot{u}_x + a\ddot{w}_\phi) + a^2EAu_y + EI_x\ddddot{u}_y + a^2EAu_\phi - EI_x\dddot{u}_\phi \\
&\qquad = a^3(aF_y - \dot{N}_x + EA\theta_\phi) \\
&EI_{xy}(\dddot{u}_x + a\dot{w}_\phi) + (EI_x\dddot{u}_y - a^2EA\dot{u}_y) - (a^2EA + EI_x)\ddot{u}_\phi \\
&\qquad = a^3(aF_\phi - N_x - EA\dot{\theta}_\phi) \\
&(EI_y + GJ)\ddot{u}_x + EI_{xy}(\ddot{u}_y - \dot{u}_\phi) + a(EI_y w_\phi - GJ\ddot{w}_\phi) = a^3N_\phi
\end{aligned} \tag{21}$$

SOLUTION OF EQUATIONS

In this section the governing equations of the previous section are specialized for the different modes of response treated, and the corresponding methods of solution are presented.

Linear Asymmetric Response (SRA 100)

This program solves linearized versions of the shell and ring equations subject to harmonic mechanical and thermal loads. Since all load and response variables are periodic functions of a ϕ with period 2π, they may be represented as Fourier series in the form $\sum_{n=0}^{\infty}(A_n \cos n\phi + B_n \sin n\phi)$, where the harmonic amplitudes A_n and B_n are in general functions of s.

If the loads have an axial plane of symmetry, say $\phi = 0$, then the Fourier series for each load term reduces to a sine or cosine series. A symmetrical loading is defined as one for which the expansions for X_1, $X_3, L_2, \theta_1, \theta_2, F_x, F_y, N_\phi, \theta_\phi$ (denoted henceforth as normal type load variables) are cosine series, whereas the expansions for the remaining loads X_2, L_1, θ_{12} F_ϕ, N_x, N_y (denoted henceforth as shear type load variables) are sine series.* The reverse is true in the case of an antisymmetrical loading, and a general load consists of both symmetric and antisymmetric components (table I).

A symmetrical response is defined as one for which the expansions for $P, Q, M_1, \xi, \eta, \chi, u_x, u_y, w_\phi$ (denoted henceforth as normal type response variables) are cosine series, whereas the expansions for S, v, and u_ϕ (denoted henceforth as shear type response variables) are sine series. The reverse is true in the case of an antisymmetric response, and a general response consists of both symmetric and antisymmetric components (table I).

Inspection of the nonlinear shell and ring equations shows that a symmetric loading gives rise only to a symmetric response. For the linearized equations, it is also true that an antisymmetric loading gives rise only to an antisymmetric response. Furthermore, for the linearized equations, the response to each load harmonic is a pure harmonic of the same wave number.

Symmetric load-response equations.- In this section the boundary-value problem for the symmetrical response to the n-th harmonic of the symmetrical load components is formulated. For the sake of simplicity, the same symbols as used previously for physical load and response variables will be used to denote the corresponding harmonic amplitudes.

Differential equations: Substitution of the symmetric load and response components for the n-th harmonic into the linearized form of equations (1) and (5) gives the following ordinary differential equations.

$$
\begin{aligned}
&(rP)' + n(r/R_2)S + n^2(r'/r)M_2 - 2(n/r)M_{12} + r[(r/R_2)X_1 - r'X_3] \\
&\qquad + nr'L_1 = 0 \\
&(rQ)' + nr'S - (n^2/R_2)M_2 - T_2 + r[r'X_1 + (r/R_2)X_3] - n(r/R_2)L_1 = 0 \\
&(rS)' + r'S - (n/R_2)M_2 - nT_2 + rX_2 - (r/R_2)L_1 = 0 \\
&(rM_1)' + r[r'P - (r/R_2)Q] - r'M_2 + 2nM_{12} + rL_2 = 0
\end{aligned}
\qquad (22a)
$$

*It will be convenient in the remaining discussion to refer to free thermal strains $\theta_1, \theta_2, \theta_{12}$, and θ_ϕ as loads. More precisely, the thermal loads are given in terms of the free themal strains by equations (9) for the shell and as $EA\theta_\phi$ for rings.

$$\begin{aligned}
&\xi' - r'\chi - (r/R_2)e_1 = 0 \\
&\eta' + (r/R_2)\chi - r'e_1 = 0 \\
&v' - (n/R_2)\xi - (r'/r)(v + n\eta) - e_{12} = 0 \\
&\chi' - \kappa_1 = 0
\end{aligned} \tag{22b}$$

In these linearized equations, the elastic foundation loads, equations (3), are considered to be included in the load terms with the given applies loads; however, the live load terms, equations (4), are neglected.

The supplemental equations, expressing the excess variables of equations (22) and (3) in terms of the eight basic variables, are equations (6), (7), and (8) with nonlinear terms omitted, plus the following from equations (11)

$$u = (r/R_2)\xi + r'\eta \tag{23a}$$

$$w = -r'\xi + (r/R_2)\eta \tag{23b}$$

$$T_1 = (r/R_2)P + r'Q \tag{23c}$$

$$re_2 = \eta + nv \tag{23d}$$

$$r\kappa_2 = r'(\chi - n^2\xi/r) + n(n\eta + v)/R_2 \tag{23e}$$

$$r(\kappa_{12} - e_{12}/R_2) = n(\xi/r - \chi) \tag{23f}$$

$$\psi = (n\eta + v)/R_2 - nr'\xi/r \tag{23g}$$

Equations (22) and the supplemental equations are a system of eight first-order differential equations which may be written compactly in vector form as

$$\underset{\sim}{Y}' = \underset{\sim}{F}(s,\underset{\sim}{Y}) \tag{24}$$

where $\underset{\sim}{Y}$ is the eight-element column vector $(P,Q,S,M_1,\xi,\eta,v,\chi)$ [fig. 2(a)].

Boundary conditions: Branch edges, branch points, the closure point of a closed branch, and the location of interior rings or other meridional discontinuities are defined as boundaries. Additional artificial points of subdivision of the meridian may be required to limit subinterval length so that the small difference of large numbers does not occur in the superposition of complementary and particular solutions of equations (24) (see p. 22). In general, the region of integration of equations (24) consists of a main branch and subsidiary branches. The main branch is a continuous line consisting of segments of the shell reference meridian which in the case of only open branches begins at some arbitrary edge and terminates at some other arbitrary edge. If the meridian contains a closed branch (only one is allowed), the closed branch is the main branch, which begins at some arbitrary nonbranching point and terminates at the same point. At a branch point, only one branch exits the branch point, i.e., has increasing s-values away from the branch point. All other branches intersecting the branch point must enter the branch point, i.e., have s-values increasing towards the branch point. All branches entering a branch point are described by increasing s before the exiting branch.

General linear boundary conditions for each boundary may be written in the form

$$[B]\Delta\{y\} + [D]\{z\} = \{L\} \tag{25}$$

where $\{y\}$ and $\{z\}$ are 4x1 force and displacement subvectors of $\underset{\sim}{Y}$ and

$$\Delta\{y\} = \pm\{y\} \qquad \text{at edges} \tag{26a}$$

$$\Delta\{y\} = \{y\}^{+} - \sum \{y\}^{-} \qquad \text{at interior boundaries} \tag{26b}$$

In equation (26a), the minus sign applies only at the terminal edge (if one exists) of the main branch; in equation (26b), $\{y\}^{+}$ is the value of $\{y\}$ at the boundary on the exiting branch, and $\Sigma\{y\}^{-}$ is the sum of the values of y on the branches entering the boundary. As implied by the form of equation (25), at interior boundaries the displacement vector $\{z\}$ is continuous, i.e.

$$\{z\}^{+} = \{z\}^{-} \tag{27}$$

The matrices [B], [D], and {L} are generated by SRA 100 in the case of force-free or ring boundaries or dome closure edges. The first two types are discussed here, whereas dome closures are discussed in Appendix B. At force-free boundaries $\Delta\{y\} = \{0\}$, so that

$$\begin{aligned} [B] &= [I] \\ [D] &= [0] \\ \{L\} &= \{0\} \end{aligned} \tag{28}$$

Substitution of the symmetric load and response components for the n-th harmonic into the linearized ring equations (21) yields

$$[k]\{u\} = \{\ell_f\} + \{\ell_t\} \tag{29}$$

where [k] is the ring stiffness matrix given by

$$[k] = \frac{1}{a}\begin{bmatrix} n^2(n^2EI_y+GJ)/a^2 & n^4EI_{xy}/a^2 & n^3EI_{xy}/a^2 & -n^2(EI_y+GJ)/a \\ & EA+n^4EI_x/a^2 & n(EA+n^2EI_x/a^2) & -n^2EI_{xy}/a \\ & & n^2(EA+EI_x/a^2) & -nEI_{xy}/a \\ \text{Symmetric} & & & EI_y + n^2GJ \end{bmatrix} \tag{30a}$$

and $\{u\}$, $\{\ell_f\}$ and $\{\ell_t\}$ are ring centroidal displacement, and mechanical and thermal load vectors given by (see fig. 3)

$$\{u\} = \begin{Bmatrix} u_x \\ u_y \\ u_\phi \\ w_\phi \end{Bmatrix} \tag{30b}$$

$$\{\ell_f\} = \begin{Bmatrix} aF_x + nN_y \\ aF_y - nN_x \\ aF_\phi - N_x \\ aN_\phi \end{Bmatrix} \tag{30c}$$

$$\{\ell_t\} = EA\theta_\phi \begin{Bmatrix} 0 \\ 1 \\ n \\ 0 \end{Bmatrix} \tag{30d}$$

In order to derive boundary conditions of the form of equation (25) from equation (29), it is necessary to relate the ring centroidal displacement and load vectors, $\{u\}$ and $\{\ell_f\}$, to the shell reference surface displacement and force jump vectors, $\{z\}$ and $\Delta\{y\}$, at the corresponding boundary point. Equilibrium of forces and moments at the ring centroid gives, in terms of the ring eccentricities e_x and e_y [fig. 3(a)],

$$\{\ell_f\} = [B]\Delta\{y\} + \{\ell_f^{(e)}\} \tag{31}$$

where

$$[B] = \frac{r}{a}\begin{bmatrix} a & 0 & -ne_x & 0 \\ 0 & a & -ne_y & 0 \\ 0 & 0 & r & 0 \\ -ae_y & ae_x & 0 & a \end{bmatrix} \tag{32}$$

and $\{\ell_f^{(e)}\}$ is given by equation (30c) with the ring forces and moments per unit of circumferential length replaced by corresponding external forces and moments. Furthermore, assuming that the ring centroid is connected to the corresponding boundary point on the shell reference meridian by a rigid link with the ring free thermal strain θ_ϕ, one obtains the kinematic relation

$$\{u\} = [e]\{z\} + \{\ell_e\} \tag{33}$$

where $[e]$ is an eccentricity matrix given by

$$[e] = \begin{bmatrix} 1 & 0 & 0 & e_y \\ 0 & 1 & 0 & -e_x \\ ne_x/r & ne_y/r & a/r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{34a}$$

$$\{\ell_e\} = \theta_\phi \begin{Bmatrix} e_x \\ e_y \\ 0 \\ 0 \end{Bmatrix} \tag{34b}$$

Substitution of equations (31) and (33) into equation (29) yields the desired boundary condition in the form of equation (25), where [B] is given by equation (32) and

$$[D] = -[k][e] \tag{35a}$$

$$\{L\} = -\{\ell_f^{(e)}\} - \{\ell_t\} + [k]\{\ell_e\} \tag{35b}$$

As a check of this result, it is noted that $[B]^{-1} = [e]^T/r$. Multiplication of equation (25) by $[B]^{-1}$ shows that the self-adjointness (i.e. symmetry) of equations (29) is preserved.

Method of solution.- The original method of solution of the linear boundary-value problem, equations (24), (25) and (27), is denoted here as the Gaussian elimination method (ref. 9). This method consists of subdividing the range of integration (i.e., the shell meridian) into a number of suitably small subintervals, the end points of which have been denoted as boundaries in the previous section. A forward integration scheme, such as Runge-Kutta, is used to integrate equation (24) over each subinterval between consecutive boundaries, to obtain eight linearly independent complementary solution vectors $\tilde{Y}_c^{(k)}$, k = 1,...,8, and a particular solution vector $\tilde{Y}_p$. Initially, at the starting point of each subinterval, the matrix of column vectors $[\tilde{Y}_c^{(k)}]$ is chosen to be the 8x8 identity matrix and $\tilde{Y}_p$ the 8x1 null matrix. The boundary conditions (25) and (27) are used to set up a system of algebraic equations for the constants of superposition for each subinterval. These are solved efficiently by Gaussian elimination in terms of 4x4 matrices, and the results used to superpose the complementary and particular solutions to obtain the desired solution. The subintervals must be small enough so that the superposition of solutions does not involve taking the small differences of large numbers, with a consequent loss of significance. It is characteristic of this method that the information contained in the boundary conditions is not used during the forward integration of the differential equations and that the initial conditions used for the complementary and particular solutions are arbitrary within the condition that the 8x8 initial value matrix $[\tilde{Y}_c^{(k)}]$ should be nonsingular.

Later, Zarghamee and Robinson (ref. 7) proposed the use of starting conditions for the complementary and particular solutions which imply satisfaction of the boundary conditions. Since four conditions are known at the initial edge, they reasoned that only four complementary solutions are required to satisfy the four conditions at the final edge. On the other hand, since only four conditions are known in terms of eight variables at the initial edge, there is still some arbitrariness in the determination of the starting conditions in this method.* Their technique

*As a consequence, the problem of "long subintervals" noted above remains in the Zarghamee method. It is noted here that a new method, termed the field method (ref. 17), which eliminates this problem as well as providing other benefits, is currently being investigated.

was generalized to general linear boundary conditions and open branched shells by Anderson, et al. (ref. 2, Appendix A). During the course of the present study, it was found that the supplemental starting conditions proposed in reference 2 lead to the inversion of poorly conditioned matrices. In this section the Zarghamee method is presented with new supplemental starting conditions, and it is also generalized to include shells with a closed branch.

Open branches: The 8x4 matrix of the four complementary solution vectors $[\underset{\sim}{Y}_c^{(k)}]$, $k = 1,\ldots,4$, and the 8x1 particular solution matrix $\underset{\sim}{Y}_p$ are partitioned into 4x4 submatrices U,W and 4x1 submatrices G,J as shown below (for simplicity in writing, the brackets and braces used for 4x4 and 4x1 matrices, respectively, are omitted in the remainder of this section).

$$[\underset{\sim}{Y}_c^{(k)}] = \begin{bmatrix} U \\ \hline W \end{bmatrix} \tag{36a}$$

$$\underset{\sim}{Y}_p = \begin{Bmatrix} G \\ \hline J \end{Bmatrix} \tag{36b}$$

Then the desired solution y,z may be written for the i-th subinterval as

$$y = G + Uc_i \tag{37a}$$

$$z = J + Wc_i \tag{37b}$$

where c_i is a 4x1 matrix of superposition constants for the i-th subinterval.

At a starting edge of an open branch, the boundary conditions (25) may be written

$$By_0 + Dz_0 = L \tag{38}$$

where the subscript 0 denotes initial values. Substitution of equations (37) into equation (38) shows that equation (38) will be satisfied regardless of the value of c for the subinterval considered if

$$BU_0 + DW_0 = 0 \tag{39a}$$

$$BG_0 + DJ_0 = L \tag{39b}$$

Equations (39) are then starting conditions for the matrices, U,W,G,J, at an initial edge.

Equations (39a) and (39b) are respectively 16 equations in 32 unknowns and 4 equations in 8 unknowns and hence do not have a unique solution. In order to formalize the procedure, it is necessary to augment equations (39) with supplementary conditions such that

(1) the initial values U_0,W_0,G_0,J_0 are uniquely determined, and

(2) the complementary solution vectors $\underset{\sim}{Y}_c^{(k)}$ are linearly independent.

In Appendix C, it is shown that condition 2 will be satisfied for any supplementary condition for U_0,W_0 of the form $\alpha U_0 + \beta W_0 = I$, where α and β are 4x4 matrices. In order to minimize the calculation of the initial values, for any particular choice of α and β, it will be convenient to choose the supplemental conditions for G_0,J_0 as $\alpha G_0 + \beta J_0 = 0$. If B is nonsingular, as is the case if no kinematic constraints are specified at the boundary, a suitable set of supplemental conditions are obtained by simply taking $\alpha = 0$ and $\beta = I$, viz.

$$W_0 = I \tag{40a}$$

$$J_0 = 0 \tag{40b}$$

Substitution of equations (40) into equations (39) gives

$$U_0 = -B^{-1}D \tag{41a}$$

$$G_0 = B^{-1}L \tag{41b}$$

If B is singular, equations (40) are replaced by

$$\pm U_0 + SW_0 = I \tag{42a}$$

$$\pm G_0 + SJ_0 = 0 \tag{42b}$$

where S is a diagonal scaling matrix, the purpose of which is to provide dimensional homogeniety to equations (42). The first three diagonal elements of S are taken to be $C_1{}^{(0)}/t$, and the fourth diagonal element is $C_1{}^{(2)}/t$, where t is an effective thickness given by

$$t = [12C_1{}^{(2)}/C_2{}^{(0)}]^{1/2} \tag{43}$$

and $C_1{}^{(0)}$ and $C_1{}^{(2)}$ are meridional stretching and bending stiffnesses [see eq. (A-3) of Appendix A].

Substitution of equations (42) into equations (39) gives the initial values

$$W_0 = (BS \mp D)^{-1}B \tag{44a}$$

$$U_0 = \pm(I - SW_0) \tag{44b}$$

$$J_0 = \mp(BS \mp D)^{-1}L \tag{44c}$$

$$G_0 = \mp SJ_0 \tag{44d}$$

Since, conceivably, either BS + D or BS - D may be singular, both upper and lower signs in equations (42) and (44) are allowed. Equations (44) could of course be used as initial edge starting values in all cases; however, the relative simplicity of equations (40) and (41) suggests their use in the common case of nonsingular B.

At an interior boundary, at which several open branches may intersect, the boundary conditions (25) and (27) may be written as (see fig. 4)

$$B[y_{i_J+1,0} - \sum_{j=1}^{J} y_{i_j,1}] + Dz_{i_J+1,0} = L \tag{45a}$$

$$z_{i_J+1,0} = z_{i_j,1} \quad , \quad j = 1,\cdots,J \tag{45b}$$

where i_j are the subinterval numbers (generally nonconsecutive) of subintervals terminating at the boundary, the number of which is denoted as J. In equations (45) the first subscript refers to the subinterval number and the second subscript 0 or 1 indicates evaluation at either the beginning or end of the subinterval, respectively. Substitution of equation (37b) into (45b) gives the c_{i_j} for entering subintervals in terms of c_{i_J+1} for the exiting subinterval, viz.*

$$c_{i_j} = W_{i_j,1}^{-1}(J_{i_J+1,0} - J_{i_j,1} + W_{i_J+1,0}c_{i_J+1}) \quad , \quad j = 1,\cdots,J \tag{46}$$

Substitution of equations (37) into equation (45a) and elimination of c_{i_j} through use of equations (46), shows that equation (45a) will be satisfied regardless of the value of c_{i_J+1} if

*It is shown in Appendix C that $W_{i_j,1}$ are nonsingular matrices.

$$BU_{i_J+1,0} + \tilde{D}W_{i_J+1,0} = 0 \tag{47a}$$

$$BG_{i_J+1,0} + \tilde{D}J_{i_J+1,0} = \tilde{L} \tag{47b}$$

where

$$\tilde{D} = D - B\sum_{j=1}^{J} \Delta_{i_j} \tag{48a}$$

$$\tilde{L} = L + B\sum_{j=1}^{J} \Lambda_{i_j} \tag{48b}$$

and

$$\Delta_{i_j} = U_{i_j,1}W_{i_j,1}^{-1} \tag{49a}$$

$$\Lambda_{i_j} = G_{i_j,1} - \Delta_{i_j}J_{i_j,1} \tag{49b}$$

Equations (47) are the starting conditions for the matrices U,W,G,J on the exiting branch of an interior boundary. Since they are of the same form as equations (39), the starting values of these matrices are also given by equations (40) and (41), or (44), with D and L replaced by $\tilde{D}$ and $\tilde{L}$, respectively.*

If the shell contains no closed branch, a terminal edge will be reached at the end of, say, the m-th subinterval. For this boundary, the boundary condition (25) may be written as

$$-By_{m,1} + Dz_{m,1} = L \tag{50}$$

---

*It may be noted here that since the starting conditions for the complementary solution matrices U,W are independent of the boundary load vectors L, U and W are independent of all load (nonhomogeneous) terms in both differential equations and boundary conditions. (The same will be seen to be true for the additional complementary solution matrices V,Z required on closed branches). Consequently, in a sequence of problems in which only load terms change, the complementary solutions need be computed just once.

Substituting equations (37) into equation (50) and solving for c_m gives

$$c_m = (DW_{m,1} - BU_{m,1})^{-1}(L - DJ_{m,1} + BG_{m,1}) \tag{51}$$

Starting with this value of c_m, equation (46) is used recursively to obtain the c_i for each subinterval, after which the solution is given by equations (37).

Closed branches: The present method requires the calculation of four additional complementary solution vectors on a closed branch. If the matrix of these vectors is partitioned into 4x4 submatrices V and Z, the desired solution may be written for the i-th subinterval on a closed branch as [cf. eqs. (37)]

$$y = G + Uc_i + Vd_i \tag{52a}$$

$$z = J + Wc_i + Zd_i \tag{52b}$$

where d_i is an additional 4x1 matrix of superposition constants for the i-th subinterval.

Equations (45) are the proper boundary conditions for an interior boundary of closed branch. Since only one closed branch is allowed and this must be chosen as the main branch (see p. 19), it follows that in this case, subintervals i_1 and i_J+1 are entering and exiting <u>closed</u> branch subintervals, and i_j , $j = 2,\cdots,J$, are entering <u>open</u> branch subintervals. Substitution of equations (52) into equations (45) shows that equations (45) will be identically satisfied with respect to c_{i_j} $(j = 1,\cdots,J+1)$, d_{i_1}, and d_{i_J+1} if c_{i_1} is given by equation (46) with $j = 1$, equations (47) are satisfied, and in addition*

$$d_{i_1} = Z_{i_1,1}^{-1} Z_{i_J+1,0}\, d_{i_J+1} \tag{53a}$$

$$c_{i_j} = W_{i_j,1}^{-1}(J_{i_J+1,0} - J_{i_j,1} + W_{i_J+1,0}\,c_{i_J+1} + Z_{i_J+1,0}\,d_{i_J+1}) , \quad j = 2,\cdots,J \tag{53b}$$

$$BV_{i_J+1,0} + \hat{D}Z_{i_J+1,0} = 0 \tag{54}$$

*It is shown in Appendix C that $Z_{i_j,1}$ are nonsingular matrices.

where

$$\hat{D} = D - B(V_{i_1,1}Z_{i_1,1}^{-1} + \sum_{j=2}^{J} \Delta_{i_j}) \tag{55}$$

In Appendix C it is shown that eight linearly independent complementary solution vectors (in subinterval i_J+1) satisfying equations (47a) and (54) do not exist if the corresponding boundary condition matrix B is singular. Therefore, singular B matrices (i.e., kinematric constraint) are not allowed at boundaries on a closed loop except at the closure (terminal) point. Furthermore, in Appendix C it is shown that the linear independence of the eight complementary solution vectors $\underset{\sim}{Y}_c^{(k)}$ depends on the non-singularity of the initial values of W and Z. Therefore, the supplemental conditions for equations (47) and (54) on a closed branch are always chosen to be

$$W_{i_J+1,0} = Z_{i_J+1,0} = I \tag{56a}$$

$$J_{i_J+1,0} = 0 \tag{56b}$$

Substitution of equations (56) into equations (47) and (54) then gives the remaining initial values

$$\begin{aligned} U_{i_J+1,0} &= -B^{-1}\tilde{D} \\ V_{i_J+1,0} &= -B^{-1}\hat{D} \\ G_{i_J+1,0} &= -B^{-1}\tilde{L} \end{aligned} \tag{57}$$

The integration on a closed branch is started at an arbitrary (nonbranching) point with the initial values

$$U_{1,0} = Z_{1,0} = I \tag{58a}$$

$$V_{1,0} = W_{1,0} = 0 \tag{58b}$$

$$G_{1,0} = J_{1,0} = 0 \tag{58c}$$

The solution for these matrices is continued by forward integration from the initial point to the final (closure) point. At intervening boundaries on the closed branch, the integration is restarted with the initial conditions given by equations (56) and (57); at intervening boundaries on open branches, the initial conditions are given by equations (40) and (41), or (44) for edge boundaries, and these same equations with D and L replaced by $\tilde{D}$ and $\tilde{L}$ for interior boundaries. As each boundary on the closed branch is passed, equations (46) with j = 1 and (53a) are applied to generate relationships giving c_1, d_1 in terms of c_k, d_k of the exiting subinterval, viz.*

$$c_1 = p_k c_k + q_k \tag{59a}$$

$$d_1 = \hat{p}_k d_k \tag{59b}$$

In view of equations (56), equations (46) for j = 1 and (53a) reduce to (setting $i_1 = k$ and $i_J + 1 = k + 1$)

$$\left.\begin{aligned} c_k &= W_{k,1}^{-1}(c_{k+1} - J_{k,1}) \\ d_k &= Z_{k,1}^{-1} d_{k+1} \end{aligned}\right\} \tag{60}$$

Substitution of equations (60) into equations (59) and comparison with equations (59) with k replaced by k + 1 gives the recursion relations

$$\left.\begin{aligned} p_{k+1} &= p_k W_{k,1}^{-1} \\ \hat{p}_{k+1} &= \hat{p}_k \, Z_{k,1}^{-1} \\ q_{k+1} &= q_k - p_k W_{k,1}^{-1} J_{k,1} \end{aligned}\right\} \tag{61}$$

which are used with the initial values

$$\left.\begin{aligned} p_1 &= \hat{p}_1 = I \\ q_1 &= 0 \end{aligned}\right\} \tag{62}$$

to generate p_k, $\hat{p}_k$, and q_k. When the final subinterval is reached, the matrices $p = p_K$, $\hat{p} = \hat{p}_K$, and $q = q_K$ will have been obtained.*

At the closure point the boundary conditions (25) and (27) may be

---

*Here, k is an index for subintervals on the closed branch only, $k = 1, \cdots, K$.

written as [cf. eq. (50)]

$$
\begin{aligned}
B(y_{1,0} - y_{m,1}) + Dz_{1,0} &= L \\
z_{1,0} &= z_{m,1}
\end{aligned}
\tag{63}
$$

Substitution of equations (52) and (58) into equations (63), and elimination of c_1 and d_1 through use of equations (59) with $k = K$ gives the following two equations for c_m and d_m.

$$
\begin{aligned}
B(p - U_{m,1})c_m + (Dp' - BV_{m,1})d_m &= L + B(G_{m,1} - q) \\
W_{m,1}c_m + (Z_{m,1} - \hat{p})d_m &= -J_{m,1}
\end{aligned}
\tag{64}
$$

The solution of equations (64) is

$$c_m = (\bar{D}W_{m,1} - B\bar{U}_{m,1})^{-1}(L - \bar{D}J_{m,1} + B\bar{G}_{m,1}) \tag{65a}$$

$$d_m = (\hat{p} - Z_{m,1})^{-1}(W_{m,1}c_m + J_{m,1}) \tag{65b}$$

where

$$
\begin{aligned}
\bar{D} &= (D\hat{p} - BV_{m,1})(\hat{p} - Z_{m,1})^{-1} \\
\bar{U}_{m,1} &= U_{m,1} - p \\
\bar{G}_{m,1} &= G_{m,1} - q
\end{aligned}
\tag{66}
$$

It may be noted that equation (65a) for c_m is of the same form as that for the open branch [eq. (51)]. Starting with these values for c_m and d_m, equations (60) are used to obtain c_i and d_i for closed branch subintervals, and equations (53b) and (46) are used to obtain c_i for open branch subintervals. The solution on the closed branch is then given by equations (52) and on open branches by equations (37).*

<u>Antisymmetric loading</u>.- As has been noted on page 17, for the linearized shell and ring equations, the response to antisymmetric load components is also antisymmetric. It is shown in this section that the antisymmetric response can be obtained from the solution of the symmetric load-response equations, outlined in the preceding sections.

---

*The equations used for the calculation of the components of the three-dimensional stress tensor from the solution for the y and z vectors are given in Appendix D.

Considering as typical normal and shear type loads the normal pressure X_3 and the circumferential shear X_2, respectively, one has

$$X_3 = \sum_{n=0}^{\infty} [X_{3_n}{}^{(s)} \cos n\phi + X_{3_n}{}^{(a)} \sin n\phi]$$
$$X_2 = \sum_{n=0}^{\infty} [X_{2_n}{}^{(s)} \sin n\phi + X_{2_n}{}^{(a)} \cos n\phi] \tag{67}$$

Here the superscripts (s) and (a) refer to symmetric and antisymmetric components, respectively. Considering as typical normal and shear type response variables the meridional stress resultant T_1 and the shear stress resultant T_{12}, respectively, one has

$$T_1 = \sum_{n=0}^{\infty} [T_{1_n}{}^{(s)} \cos n\phi + T_{1_n}{}^{(a)} \sin n\phi]$$
$$T_{12} = \sum_{n=0}^{\infty} [T_{12_n}{}^{(s)} \sin n\phi + T_{12_n}{}^{(a)} \cos n\phi] \tag{68}$$

From the identities

$$\sin n\phi = \cos(n\phi - \pi/2)$$
$$\cos n\phi = -\sin(n\phi - \pi/2) \tag{69}$$

if follows that the antisymmetric load components are equivalent to symmetric components about a rotated plane according to

$$X_{3_n}{}^{(a)} \sin n\phi = X_{3_n}{}^{(a)} \cos(n\phi - \pi/2)$$
$$X_{2_n}{}^{(a)} \cos n\phi = -X_{2_n}{}^{(a)} \sin(n\phi - \pi/2) \tag{70}$$

Therefore, using the amplitudes $X_3 = X_{3_n}{}^{(a)}$ and $X_2 = -X_{2_n}{}^{(a)}$ in the symmetric response equations will give the amplitudes T_1 and T_{12} corresponding to the solution

$$T_1 \cos(n\phi - \pi/2) = T_1 \sin n\phi$$
$$T_{12} \sin(n\phi - \pi/2) = -T_{12} \cos n\phi \tag{71}$$

Comparison of the right hand sides of equations (71) with equations (68) shows that

$$T_{1_n}^{(a)} = T_1 \qquad (72)$$
$$T_{12_n}^{(a)} = -T_{12}$$

One therefore concludes that the same equations used for the symmetric components can be used also to obtain the antisymmetric components if the signs of the shear type antisymmetric load amplitudes are reversed before the solution and the signs of the shear type antisymmetric response amplitudes are reversed after the solution. Hence, the complementary solutions and the [B],[D] boundary matrices for the symmetric response components of a particular harmonic can be used also for the antisymmetric response components of the same harmonic.

Buckling of Axisymmetric Equilibrium States (SRA 101)

This program calculates bifurcation buckling modes of linearized asymmetric prebuckling states. The structural loading is assumed to have a given spatial distribution, but its magnitude is allowed to vary in proportion to a load parameter λ. This leads to a set of eigenvalue equations for the critical load λ_c. A general form of the eigenvalue equations for bifurcation buckling and their method of solution are presented in Appendix E. In the development presented there, no restrictions are placed on the structural geometry or loading, and nonlinear prebuckling states are included. In this section the iteration equations (E-11) and the inner product [eq. (E-6), required to calculate the eigenvalue estimate according to the Rayleigh quotient, eq. (E-19)] are specialized to ring-stiffened shells of revolution assuming linearized prebuckling states and neglecting prebuckling rotations.

Iteration equations.- For numerical purposes, the differential form of the variational iteration equations (E-11) is more convenient to use and is derived here. First, the differential form of the variational eigenvalue equations (E-4), from which the iteration equations follow, is obtained. Applying the usual procedure of taking the difference of the governing shell and ring equations evaluated for an initial (prebuckling) equilibrium state and an adjacent (buckling) equilibrium state, and linearizing the resulting equations for the perturbation variables, leads to the buckling eigenvalue equations. The nonlinear shell and ring equations have been presented earlier in the form of linear differential equations with nonlinear (and live load) terms isolated as additional effective mechanical and thermal loads given by equations (2), (4), (12), (13), (14), and (20). Hence, the eigenvalue equations are obtained in the same form as the linear system of equations [with dead load terms dropped from the linearized form of eqs. (1), (7), (8) and (21)] with effective additional loads derived from equations (2), (4), (12), (13), (14), and (20). Since, however, prebuckling rotations are neglected in this analysis, the contributions from equations (12), (13), (14), and (20c) are not retained. This leads to the following additional effective loads: from equations (2)

$$
\begin{aligned}
X_1 &= -\lambda[(\bar{T}_1 + \bar{T}_2)\theta]^{\cdot}/r \\
X_2 &= -\lambda(r'/r)(\bar{T}_1 + \bar{T}_2)\theta \\
X_3 &= 0 \\
L_1 &= \lambda(\bar{T}_2\psi + \bar{T}_{12}\chi) \\
L_2 &= -\lambda(\bar{T}_1\chi + \bar{T}_{12}\psi)
\end{aligned}
\tag{73}
$$

from equations (4), for live pressure loading,

$$
\begin{aligned}
X_1 &= \lambda p\chi \\
X_2 &= \lambda p\psi \\
X_3 &= \lambda p(e_1 + e_2) \\
L_1 &= L_2 = 0
\end{aligned}
\tag{74}
$$

which are identical in form to equations (4) except that pressure gradient terms are neglected, and from equations (20a,b)

$$
\begin{aligned}
N_x &= -\lambda\bar{T}_\phi w_x \\
N_y &= -\lambda\bar{T}_\phi w_y \\
F_x &= F_y = F_\phi = N_\phi = 0
\end{aligned}
\tag{75}
$$

In equations (73), (74), and (75), unbarred variables represent buckling mode variables, and barred variables represent unit load prebuckling state variables.*

The iteration equations are then obtained by setting $\lambda = 1$ in equations (73), (74), and (75), interpreting the unbarred variables in

---

*Note that since only linearized prebuckling states are treated, in the notation of Appendix E, $\lambda_0 = 0$, $u_0 = \sigma_0 = 0$, and $\lambda = \mu$.

these equations as being known inputs from the previous [(k - 1)-th] iteration, and solving the linear system of equations with these loads for the variables of the present (k-th) iteration.

In general, both the prebuckling and buckling variables of equations (73), (74), and (75) are represented by Fourier series in the circumferential coordinate ϕ. Since the product of two Fourier series is also a Fourier series*, it is seen that each iteration step reduces to an ordinary linear statics problem with multiharmonic loading. The solution for each component (symmetric and antisymmetric) of each harmonic of the effective loading is presented in the preceding discussion of linear asymmetric response (SRA 100). Inspection of equations (73), (74), and (75) shows that a symmetric prebuckling state results in decoupled symmetric and antisymmetric buckling modes, whereas an antisymmetric or a general prebuckling state results in a buckling mode with coupled symmetric and antisymmetric response (cf. p.17). Even in this case, however, the symmetric and antisymmetric components of each harmonic for each iteration step are calculated independently (cf. p. 30).

<u>Inner product</u>.- After each iteration step, the Rayleigh quotient [eq. (E-19)] is used to calculate the corresponding eigenvalue estimate. For this purpose, it is necessary to be able to compute inner products, defined by equation (E-6). Since prebuckling rotations are neglected, equation (E-6) reduces to

$$(u;\tilde{u}) = \sigma_0{}^{(1)}\cdot L_{11}(u,\tilde{u}) - q_1(u)\cdot\tilde{u} \tag{76}$$

Evaluation of equation (76) for moderate rotation theory of ring-stiffened shells of revolution gives

$$\begin{aligned}(u;\tilde{u}) = \int_{\phi=0}^{2\pi} \Big\{\int_s [&\bar{T}_1(\chi\tilde{\chi} + \theta\tilde{\theta}) + \bar{T}_2(\psi\tilde{\psi} + \theta\tilde{\theta}) \\ &+ \bar{T}_{12}(\chi\tilde{\psi} + \psi\tilde{\chi}) - p(\chi\tilde{u} + \psi\tilde{v} \\ &+ \{e_1 + e_2\}\tilde{w})]r ds + \sum_r a\bar{T}_\phi(w_x\tilde{w}_x + w_y\tilde{w}_y)\Big\}d\phi\end{aligned} \tag{77}$$

where u and $\tilde{u}$ are any two kinematically admissible displacement fields, the integral over s ranges over the whole meridional length of the shell,

---

*Multiplication of Fourier series is discussed further in the description of subroutine MODINT, ref. 8, p. 93.

and the summation over r ranges over all attached rings.

Comparison of equation (76) with the variational eigenvalue equation (E-4c), with terms depending on prebuckling rotations dropped, shows that the inner product is, in this case, nothing more than the work of the effective loads (for unit λ) associated with the displacement field u (or $\tilde{u}$) acting through the displacements $\tilde{u}$ (or u). This observation can be made explicit as follows. From equations (1), equivalent shell forces are defined as

$$
\begin{aligned}
F_1 &= -(r/R_2)X_1 + r'(X_3 - \dot{L_1}/r) \\
F_2 &= -r'X_1 - (r/R_2)(X_3 - \dot{L_1}/r) \\
F_3 &= -X_2 + L_1/R_2 \\
F_4 &= -L_2
\end{aligned}
\tag{78}
$$

and, from equations (21), equivalent ring forces are defined as

$$
\begin{aligned}
FL_1 &= -(aF_x + \dot{N_y}) \\
FL_2 &= -(aF_y - \dot{N_x}) \\
FL_3 &= -(aF_\phi - N_x) \\
FL_4 &= -aN_\phi
\end{aligned}
\tag{79}
$$

Substituting the expressions for u,w,θ,ψ from equations (11) and the expressions for w_x,w_y from equations (16) into equation (77), then performing integrations with respect to ϕ by parts, one obtains the alternate expression for the inner product

$$
\begin{aligned}
(u;\tilde{u}) = \int_0^{2\pi} [\int_s (F_1\tilde{\xi} + F_2\tilde{\eta} + F_3\tilde{v} + F_4\tilde{\chi})rds \\
+ \sum_r (FL_1\tilde{u}_x + FL_2\tilde{u}_y + FL_3\tilde{u}_\phi)]d\phi
\end{aligned}
\tag{80}
$$

where the equivalent forces are given by equations (78) and (79) with effective loads given by equations (73), (74), and (75) with $\lambda = 1$.

In equation (80), each of the displacements and equivalent forces are represented by a Fourier series in the circumferential coordinate ϕ.

Therefore, the integrand and summand are also Fourier series. However, because of the ϕ-integration only their axisymmetric components contribute to the inner product and higher harmonics may be ignored.

Nonlinear Axisymmetric Response (SRA 200)

This program solves the nonlinear large-deflection shell equations for the case of axisymmetric torsionless loading. The loading is assumed to be proportional with the load parameter λ, and the first (and possibly second) derivatives with respect to λ of the response variables [i.e., linear perturbation state(s)] at an input load level λ_0, as well as the nonlinear response at λ_0, are calculated. The numerical solution for the nonlinear state is based on a generalization of Newton's method for calculating the roots of nonlinear algebraic equations by iteration. In addition, for purely mechanical loading, the prebuckling structural stiffness K_0 (ref. 18) and its derivative $K_0^{(1)}$ at λ_0 are computed. As in reference 18, K is defined as $d\lambda/d\Delta$, where Δ is the "work deflection" defined such that the area under the $\lambda - \Delta$ curve represents the work of the external loading. As shown below, K_0 and $K_0^{(1)}$ are useful in calculating the value of a limit load λ^*, at which the Newton iteration does not converge (fig. 5).

Formulation of equations.- For axisymmetric torsionless loading, shear type load and response variables (cf. p. 17) are identically zero, and the equations associated with them (the axisymmetric torsion equations) are dropped from the system of governing equations, thereby reducing their differential order from eight to six. Since the general nonlinear equations are of the same form as the linear equations plus additional effective load terms, the differential equations are obtainable directly from equations (22) with n = 0. These are

$$
\left.
\begin{aligned}
&(rP)' + r[(r/R_2)X_1 - r'X_3] = 0 \\
&(rQ)' - T_2 + r[r'X_1 + (r/R_2)X_3] = 0 \\
&(rM_1)' + r[r'P - (r/R_2)Q] - r'M_2 + rL_2 = 0 \\
&\xi' - r'\chi - (r/R_2)e_1 = 0 \\
&\eta' + (r/R_2)\chi - r'e_1 = 0 \\
&\chi' - \kappa_1 = 0
\end{aligned}
\right\} \qquad (81)
$$

The supplemental equations are equations (7a-d), linearized by replacing $\varepsilon_1,\varepsilon_2$ by e_1,e_2, equations (8a,b), (23a-c), and from equations (23d,e)

$$e_2 = \eta/r$$
$$\kappa_2 = r'\chi/r \tag{82}$$

Effective loads, in addition to real proportional (dead) loads $\lambda\bar{X}_1$, $\lambda\bar{X}_3$, $\lambda\bar{\theta}_1$, $\lambda\bar{\theta}_2$, are from equations (2),

$$L_2 = -T_1\chi \tag{83a}$$

$$X_1 = X_3 = L_1 = 0 \tag{83b}$$

plus equation (3a,c,e), (4a,c,e)*, and from equations (12)

$$\theta_1 = -\chi^2/2 \tag{84a}$$

$$\theta_2 = 0 \tag{84b}$$

Equations (81) and its supplemental equations are a system of six first-order differential equations which are of the same form as equation (24), where now $\underset{\sim}{Y}$ is the six-element column vector (P,Q,M_1,ξ,η,χ) and $\underset{\sim}{F}$ is a nonlinear vector function of $\underset{\sim}{Y}$. Note that all of the nonlinearity is exhibited in the two effective load terms L_2 and θ_1.

The effective ring loads due to nonlinear terms [eqs.(20)] are identically zero, since the ring rotations are shear type variables. Therefore, for axisymmetric torsionless loading, the ring equations and associated boundary conditions are linear. Other boundary conditions are also assumed to be linear so that equation (25) applies. However, in this case [B] and [D] are 3x3 matrices, and the force, displacement, and load matrices, {y}, {z}, and {L}, are 3x1 matrices. For rings, equations (29)-(35) apply with n set to zero and the third row and column of each matrix deleted.

Newton's method.- Following Thurston (ref. 19) a generalization of Newton's method for differential equations is used to reduce the non-linear boundary-value problem to a sequence of linear boundary-value problems. In this method, the iteration equations are derived by assuming that the solution is given by a small correction to an approximate solution (initially taken as the linear solution) and linearizing the

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*The effect of the pressure gradient terms in equation (4c) is neglected in the program.

differential equations with respect to the correction.* Substituting the solution $\underset{\sim}{Y}_{(k)}$ after k iterations

$$\underset{\sim}{Y}_{(k)} = \underset{\sim}{Y}_{(k-1)} + \delta\underset{\sim}{Y} \tag{85}$$

into equation (24), linearizing with respect to $\delta\underset{\sim}{Y}$, and using equation (85) to eliminate $\delta\underset{\sim}{Y}$, gives the iteration equations

$$\underset{\sim}{Y}_{(k)}' - \underset{\sim}{F}_{Y}\underset{\sim}{Y}_{(k)} = \underset{\sim}{F} - \underset{\sim}{F}_{Y}\underset{\sim}{Y}_{(k-1)} \tag{86}$$

where $\underset{\sim}{F}_{Y}$ is a matrix, the (i,j) element of which is the derivative of i-th component of $\underset{\sim}{F}$ with respect to the j-th component of $\underset{\sim}{Y}$, and $\underset{\sim}{F}$ and $\underset{\sim}{F}_{Y}$ are both evaluated for $\underset{\sim}{Y} = \underset{\sim}{Y}_{(k-1)}$. Equations (86) are a linear system of equations, which, when supplemented by the boundary conditions (25), are solved by the Zarghamee method (p. 22).

For the specialized equations of the previous section, each variable can be written as in equation (85). Only the nonlinear terms given by the additional loads of equations (83a) and (84a) need be expanded, i.e.

$$\begin{aligned} L_{2(k)} &= -T_{1(k)}\chi_{(k)} = -(T_{1(k-1)} + \delta T_1)(\chi_{(k-1)} + \delta\chi) \\ &\simeq -T_{1(k-1)}\chi_{(k-1)} - T_{1(k-1)}\delta\chi - \chi_{(k-1)}\delta T_1 \\ &= -T_{1(k-1)}\chi_{(k)} - \chi_{(k-1)}T_{1(k)} + T_{1(k-1)}\chi_{(k-1)} \end{aligned} \tag{87}$$

and similarly

$$\theta_{1(k)} = -\chi_{(k-1)}\chi_{(k)} + (1/2)\chi_{(k-1)}^2 \tag{88}$$

---

*In this method, each iteration step yields an approximate solution which satisfies the boundary conditions exactly, but the differential equations only approximately. In reference 20, an alternate form of Newton's method is proposed, whereby the unknowns to be corrected are the initial values of $\underset{\sim}{Y}$ for each subinterval. In this method, each iteration step yields an approximate solution which satisfies the differential equations exactly, but the boundary conditions only approximately.

The last term on the right-hand side of each of equations (87) and (88) are nonhomogeneous terms known from the previous iteration, whereas the remaining terms are linear in the variables T_1 and χ of the present iteration.

It may be observed that in general if $\underset{\sim}{Y}_{(k-1)}$ is an exact solution of equation (24), then the solution of equation (86) is $\underset{\sim}{Y}_{(k)} = \underset{\sim}{Y}_{(k-1)}$, as it should be in the limit as $k \to \infty$. However, in this case equations (86) represent the variational equations about the equilibrium state $\underset{\sim}{Y}_{(k-1)}$. Therefore, if the load λ_0 is at a limit load λ^*, a unique solution for $\underset{\sim}{Y}_{(k)}$ does not exist (i.e., the system becomes singular). Since a unique solution for $\underset{\sim}{Y}_{(k)}$, given an exact solution $\underset{\sim}{Y}_{(k-1)}$, is a necessary condition for convergence, clearly the iteration method diverges at a limit point, and in fact, the rate of convergence becomes impractically slow for λ_0 sufficiently close to λ^*.

In practice, however, it is not necessary to observe divergence of the method in order to estimate λ^*. This may be done using the stiffness K_0 and its derivative $K_0^{(1)}$ obtained at $\lambda_0 < \lambda^*$ (fig. 5). Since at λ^*, $K_0 = d\lambda/dK_0 = 0$, in the vicinity of λ^*, λ may be expanded approximately as

$$\lambda \simeq \lambda^* + \alpha K_0^2 \tag{89}$$

The constant α is evaluated by differentiation of equation (89) with respect to K_0 to give

$$d\lambda/dK_0 = 1/K_0^{(1)} \simeq 2\alpha K_0 \tag{90}$$

Substitution of equation (90) into equation (89) shows that as λ_0 approaches λ^*, λ^* may be computed as

$$\lambda^* \simeq \lambda_0 - K_0/2K_0^{(1)} \tag{91}$$

The evaluation of K_0 and $K_0^{(1)}$ is presented in the description of subroutine STREN in reference 8, p. 106.

<u>Linear perturbation states</u>.- The calculation of K_0 and $K_0^{(1)}$ requires the determination of not only the nonlinear response at the given load λ_0, but also the first and second derivatives of the response (with respect to λ) at λ_0.† The differential equations for these states are obtained by

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†The first derivative state is always computed since it is required input to the buckling program SRA 201.

differentiation of equation (24) with respect to λ to give

$$\underset{\sim}{Y}^{(1)'} - \underset{\sim}{F}_{\underset{\sim}{Y}}\underset{\sim}{Y}^{(1)} = 0 \tag{92a}$$

$$\underset{\sim}{Y}^{(2)'} - \underset{\sim}{F}_{\underset{\sim}{Y}}\underset{\sim}{Y}^{(2)} = \underset{\sim}{F}_{\underset{\sim}{YY}}\underset{\sim}{Y}^{(1)} \tag{92b}$$

where $\underset{\sim}{F}_{YY}$ is a matrix, the (i,j) element of which is the second derivative of the i-th component of $\underset{\sim}{F}$ with respect to the j-th component of $\underset{\sim}{Y}$, and $\underset{\sim}{F}_Y$ and $\underset{\sim}{F}_{YY}$ are both evaluated at the converged nonlinear solution $\underset{\sim}{Y}$.

The corresponding boundary conditions are equations (25) with {L} replaced by the unit load $\{\bar{L}\}$ for the first derivative state and {L} replaced by {0} for the second derivative state. Since equations (92) and the boundary conditions are linear, they are also solved by the Zarghamee method. Comparison of equations (92a,b) with (86) shows that insofar as $\underset{\sim}{Y}_{(k-1)} \simeq \underset{\sim}{Y}_{(k)} \simeq \underset{\sim}{Y}$, the homogeneous forms of these three equations are identical. Consequently, the complementary solutions obtained in the last Newton iteration may be used in the calculation of $\underset{\sim}{Y}^{(1)}$ and $\underset{\sim}{Y}^{(2)}$.

The nonhomogeneous terms of equations (92a,b) are obtained by differentiating with respect to λ the dead loads $\lambda\bar{X}_1$, $\lambda\bar{X}_3$, $\lambda\bar{\theta}_1$, $\lambda\bar{\theta}_2$, the live loads given in equations (4a,c), and the effective loads given by equations (83a) and (84a). For $\underset{\sim}{Y}^{(1)}$, this gives

$$\begin{aligned} X_1 &= \bar{X}_1 + p\chi \\ X_3 &= \bar{X}_3 + p(e_1 + e_2) + \xi\partial p/\partial x + \eta\partial p/\partial y \\ L_2 &= 0 \\ \theta_1 &= \bar{\theta}_1 \\ \theta_2 &= \bar{\theta}_2 \end{aligned} \tag{93}$$

where ξ,η,χ,e_1,e_2 are response variables of the nonlinear solution.

For $\underset{\sim}{Y}^{(2)}$, one obtains

$$
\begin{aligned}
X_1 &= 2p\chi^{(1)} \\
X_3 &= 2p(e_1{}^{(1)} + e_2{}^{(1)}) + 2\xi^{(1)}\partial p/\partial x + 2\eta^{(1)}\partial p/\partial y \\
L_2 &= -2T_1{}^{(1)}\chi^{(1)} \\
\theta_1 &= -\chi^{(1)2} \\
\theta_2 &= 0
\end{aligned}
\tag{94}
$$

Equations (92a) and (92b) are then equivalent to equations (81) and its supplemental conditions with nonhomogeneous load terms given by equations (93) and (94), respectively.

Buckling of Axisymmetric Equilibrium States (SRA 201)

This program calculates the bifurcation buckling modes of nonlinear (or linear) axisymmetric torsionless prebuckling states. The structural loading is assumed to have a given spatial distribution, but its magnitude is allowed to vary in proportion to a load parameter λ. This leads to a set of eigenvalue equations, which are linearized with respect to λ by expanding the prebuckling state variables in a Taylor series in $\mu = \lambda - \lambda_0$, and retaining only linear terms in μ.

Geometrically, this method consists of examining the stability of fictitious equilibrium states on the tangent to the nonlinear load-deformation curve at an assumed load λ_0 below the critical load. For loads near λ_0, the corresponding fictitious states are good approximations to the neighboring nonlinear states. Consequently, as λ_0 is increased towards the critical load, the fictitious critical loads obtained approximate with increasing precision the actual critical load. For each λ_0, the method of successive approximations is used to obtain the fictitious critical load.

A general form of the eigenvalue equations for bifurcation buckling and the method of solution are presented in Appendix E.* In the development presented there, no restrictions are placed on the structural geometry or loading. In this section the iteration equations (E-11) and the inner product [eq. (E-6), required to calculate the eigenvalue estimate according to the Rayleigh quotient, eq. (E-19)] are specialized to ring-stiffened shells of revolution under axisymmetric torsionless loading.

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*A more specific formulation giving additional details is presented in reference 11.

Iteration equations.- For numerical purposes, the differential form of the variational iteration equations (E-11) is more convenient to use and is derived here. First, the differential form of the variational buckling equations (E-1) and eigenvalue equations (E-4), from which the iteration equations follow, are obtained. Applying the usual procedure of taking the difference of the governing shell and ring equations evaluated for an initial (prebuckling) equilibrium state and an adjacent (buckling) equilibrium state, and linearizing the resulting equations for the perturbation variables, leads to the basic buckling equations [eqs. (E-1)]. The nonlinear shell and ring equations have been presented earlier in the form of linear differential equations with nonlinear (and live load) terms isolated as additional effective mechanical and thermal loads given by equations (2), (4), (12), (13), (14) and (20). Hence, the buckling equations are obtained in the same form as the linear system of equations [with dead load terms dropped from the linearized form of eqs. (1), (7), (8) and (21)] with effective additional loads derived from equations (2), (4), (12), (13), (14) and (20). However, as mentioned previously, the terms of equations (14) are of higher order for moderate rotations and are therefore neglected. Furthermore, it is shown in reference 11 that it is consistent with this approximation to neglect the thermal load of equation (13). For axisymmetric torsionless loading, the remaining equations give the following additional effective loads:

from equations (2),

$$X_1 = -[(T_{1_0} + T_{2_0})\theta]^{\cdot}/r \tag{95a}$$

$$X_2 = -(r'/r)(T_{1_0} + T_{2_0})\theta \tag{95b}$$

$$X_3 = 0 \tag{95c}$$

$$L_1 = T_{2_0}\psi + \chi_0 T_{12} \tag{95d}$$

$$L_2 = -(T_{1_0}\chi + \chi_0 T_1) \tag{95e}$$

from equations (4),

$$\left.\begin{aligned} X_1 &= \lambda p \chi \\ X_2 &= \lambda p \psi \\ X_3 &= \lambda[p(e_1 + e_2) + \xi \partial p/\partial x + \eta \partial p/\partial y] \\ L_1 &= L_2 = 0 \end{aligned}\right\} \tag{96}$$

from equations (12a),

$$
\begin{aligned}
\theta_1 &= -\chi_0 \chi \\
\theta_2 &= 0 \\
\theta_{12} &= -\chi_0 \psi
\end{aligned}
\tag{97}
$$

and from equations (20),

$$
\begin{aligned}
N_x &= -T_{\phi_0} w_x \\
N_y &= -T_{\phi_0} w_y \\
\theta_\phi &= F_x = F_y = F_\phi = N_\phi = 0
\end{aligned}
\tag{98}
$$

In equations (95)-(98), variables with the subscript 0 are prebuckling state variables at the load λ, and the remaining response variables are buckling mode variables.

Inspection of equations (95)-(98) shows that not only are the equations for individual harmonics of the buckling variables decoupled, but also that the symmetric and antisymmetric components of each harmonic are decoupled. Therefore the buckling equations may be written in terms of symmetric or antisymmetric harmonic amplitudes of a single buckling harmonic. Equations (22) and their associated supplemental equations apply to the amplitudes of symmetric components of the n-th harmonic of the buckling mode. The loads for these equations are given by equations (3) and (95)-(97) with equation (95a) replaced by the corresponding symmetric load amplitude

$$X_1 = -(n/r)(T_{1_0} + T_{2_0})\theta \tag{99}$$

where from equations (11)

$$\theta = (r'/r)(v + n\eta) + (n/R_2)\xi \tag{100}$$

Boundary conditions for ring boundaries are given by equations (25), (32), (35a), and (35b) with $\{\ell_t\} = \{\ell_e\} = \{0\}$ and $\{\ell_f^{(e)}\}$ given by equations (30c) and (98), where from equations (16) the symmetric rotation amplitudes for rings are

$$w_x = -(nu_y + u_\phi)/a$$
$$w_y = nu_x/a \tag{101}$$

Using the kinematic equation (33), relating shell and ring displacements, the effective load vector for this case may be written as

$$\{L\} = [\kappa][e]\{z\} \tag{102}$$

where

$$[\kappa] = (T_{\phi_0}/a)\begin{bmatrix} n^2 & 0 & 0 & 0 \\ & n^2 & n & 0 \\ & & 1 & 0 \\ \text{Symmetric} & & & 0 \end{bmatrix} \tag{103}$$

Since the equations for antisymmetric components are the same set of equations with n replaced by -n, inspection of these equations shows that antisymmetric buckling modes have the same critical value of λ as symmetric buckling modes, and therefore need not be considered.*

In order to apply the iteration method, it is necessary to search for eigenvalues in a sufficiently small neighborhood of an estimate $\lambda = \lambda_0$, so that in this neighborhood the prebuckling variables $T1_0$, $T2_0$, χ_0, and T_{ϕ_0} have a linear dependence on λ. Setting

$$\lambda = \lambda_0 + \mu \tag{104}$$

one has, to first order in μ†

$$T1_0(\lambda) \simeq T1_0 + \mu T1_0^{(1)}$$
$$T2_0(\lambda) \simeq T2_0 + \mu T2_0^{(1)}$$
$$\chi_0(\lambda) \simeq \chi_0 + \mu \chi_0^{(1)} \tag{105}$$
$$T_{\phi_0}(\lambda) \simeq T_{\phi_0} + \mu T_{\phi_0}^{(1)}$$

*Antisymmetric buckling mode shapes are derivable from the corresponding symmetric buckling mode shapes simply by changing the sign of the amplitudes of either the shear or normal type variables.

†Henceforth, variables with the subscript 0 are assumed to be evaluated at $\lambda = \lambda_0$.

The eigenvalue problem for μ is then obtained by substituting equations (104) and (105) into the effective additional load (amplitude) terms given by equations (99), (95b-e), (96), (97), and (102). Each of the effective load terms are thus split into two parts, one part independent of μ, and a second part linear in μ. The set of equations so formed corresponds to equations (E-4) of Appendix E.

The iteration equations (E-11) are obtained by setting $\mu = 1$ in the second parts of the effective load terms and interpreting the buckling mode variables of these parts as being known inputs from the previous iteration. Thus the first parts of the effective loads become homogeneous terms and the second parts become nonhomogeneous terms for the equivalent linear problem of each iteration.

<u>Inner product</u>.- After each iteration step, the Rayleigh quotient [eq. (E-19)] is used to calculate the corresponding eigenvalue estimate. For this purpose, it is necessary to be able to compute inner products, defined by equation (E-6).* Evaluation of equation (E-6) for ring-stiffened shells of revolution under axisymmetric torsionless loads gives

$$\begin{aligned}(u,\sigma;\tilde{u},\tilde{\sigma}) = \int_{\phi=0}^{2\pi} \Big\{\int_s \Big[&{T_1}_0^{(1)}(\chi\tilde{\chi} + \theta\tilde{\theta}) + {T_2}_0^{(1)}(\psi\tilde{\psi} + \theta\tilde{\theta}) \\ &+ \chi_0^{(1)}(T_1\tilde{\chi} + \tilde{T}_1\chi + T_{12}\tilde{\psi} + \tilde{T}_{12}\psi) - p\chi\tilde{u} - p\psi\tilde{v} - (p\{e + e_2\} \\ &+ \xi\partial p/\partial x + \eta\partial p/\partial y)\tilde{w}\Big]r\,ds + \sum_r a{T_\phi}_0^{(1)}(w_x\tilde{w}_x + w_y\tilde{w}_y)\Big\}d\phi \qquad (106)\end{aligned}$$

where u and $\tilde{u}$ are any two kinematically admissible displacement fields (σ and $\tilde{\sigma}$ being corresponding stress states), the integral over s ranges over the whole shell meridian, and the summation over r ranges over all attached rings. Since prebuckling variables are axisymmetric, if buckling variables are considered to be amplitudes of symmetric harmonic components, the integral over ϕ in equation (106) can be replaced by the factor π (or 2π in the case of an axisymmetric buckling mode). Before doing so, however, a more concise form of the inner product is derived from equation (106).

Comparison of equation (E-6) with equations (E-4) shows that the inner product is equivalent to the work of the second part of the effective mechanical loads and negative free thermal strains (for unit μ)

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*The inner product is also used for orthogonalizing mode estimates with respect to lower eigenmodes when calculating nonfundamental eigenvalues.

associated with the displacement and stress fields u,σ (or $\tilde{u},\tilde{\sigma}$) acting through the displacement and stress fields $\tilde{u},\tilde{\sigma}$ (or u,σ). This observation can be made explicit by using equations (78) and (79) and integrating by parts with respect to ϕ, as was done previously for SRA 101. The result is*

$$(u,\sigma;\tilde{u},\tilde{\sigma}) = \int_s (F_1\tilde{\xi} + F_2\tilde{\eta} + F_3\tilde{v} + F_4\tilde{\chi} - \tilde{T}_1\theta_1 - \tilde{T}_{12}\theta_{12})rds$$

$$+ \sum_r (FL_1\tilde{u}_x + FL_2\tilde{u}_y + FL_3\tilde{u}_\phi) \qquad (107)$$

where the equivalent forces are given by equations (78) and (79) with $L_1{}^{\bullet}$, $N_x{}^{\bullet}$, and $N_y{}^{\bullet}$ replaced by their symmetric components nL_1, nN_x, and nN_y, respectively, and effective loads given by equations (99), (95b-e), (96), (97) and (98), with λ replaced by unity and prebuckling quantities replaced by derivatives with respect to λ at λ_0. In equation (107) the factor π (or 2π for axisymmetric buckling) representing the integration with respect to ϕ has been dropped.

Vibrations About Axisymmetric Equilibrium States (SRA 300)

This program calculates free vibration modes about nonlinear axisymmetric torsionless equilibrium states. The eigenvalue equations for the square frequency ω^2 of harmonic vibrations about an equilibrium state are similar in structure to the eigenvalue equations for the critical load increment μ for buckling in the vicinity of the same equilibrium state. As such they are solved by the same method of successive approximations as discussed previously for SRA 201. In this section the iteration equations solved and the inner product, used for calculation of the sequence of eigenvalue estimates and also for mode orthogonalization when obtaining nonfundamental modes, are presented.

Iteration equations.- The eigenvalue equations for vibrations about an equilibrium state are obtained from the eigenvalue equations for buckling in the vicinity of the same equilibrium state by replacing the incremental loads proportional to μ in the buckling problem by the inertial loads proportional to ω^2 of harmonic vibrations. Based on the thin shell assumption that each normal element acts as a rigid body with five degrees of freedom, the timewise amplitudes of shell inertial loads due to harmonic

---

*Equation (107) differs in sign (which is immaterial) from the inner product as given in reference 11.

vibrations of frequency ω are

$$
\begin{aligned}
X_1 &= \omega^2[m^{(0)}u + m^{(1)}\chi] \\
X_2 &= \omega^2[m^{(0)}v + m^{(1)}\psi] \\
X_3 &= \omega^2 m^{(0)}w \\
L_1 &= -\omega^2[m^{(2)}\psi + m^{(1)}v] \\
L_2 &= \omega^2[m^{(2)}\chi + m^{(1)}u]
\end{aligned}
\tag{108}
$$

where the mass moments per unit of surface area are given by the integrals through the shell wall thickness

$$m^{(i)} = \int \rho z^i dz \qquad \text{for } i = 0, 1, \text{ or } 2 \tag{109}$$

Similarly for homogeneous rings, assuming that each cross section acts as a rigid element, one obtains

$$
\begin{aligned}
F_x &= \omega^2\rho A u_x \\
F_y &= \omega^2\rho A u_y \\
F_\phi &= \omega^2\rho A u_\phi \\
N_x &= \omega^2\rho(I_x w_x - I_{xy}w_y) \\
N_y &= \omega^2\rho(I_y w_y - I_{xy}w_x) \\
N_\phi &= \omega^2\rho(I_x + I_y)w_\phi
\end{aligned}
\tag{110}
$$

As previously discussed for the buckling equations, the eigenvalue equations need only be written for the symmetric components of each circumferential harmonic, in which case equation (99) replaces equation (95a). The corresponding boundary conditions for ring boundaries are given by equations (25), with [B] and [D] defined by equations (32), and (35a). In addition, the effective load {L} has two components, one given by equation (102) and the other obtained from equations (35b) with $\{\ell_t\} = \{\ell_e\} = 0$ and $\{\ell_f^{(e)}\}$ determined by equations (30c) and (110). This second component may be reduced to

$$\{L\} = -\omega^2[\mu][e]\{z\} \tag{111}$$

where

$$[\mu] = \frac{\rho}{a}\begin{bmatrix} a^2A+n^2I_y & n^2I_{xy} & nI_{xy} & 0 \\ & a^2A+n^2I_x & nI_x & 0 \\ & & a^2A+I_x & 0 \\ \text{Symmetric} & & & a^2(I_x+I_y) \end{bmatrix} \tag{112}$$

The iteration equations are obtained by setting the eigenvalue parameter $\omega^2 = 1$ in equations (108) and (111) and interpreting u,v,w,χ, and ψ in equations (108) and $\{z\}$ in equation (111) as being known inputs from the previous iteration. These equations thus give the nonhomogeneous terms for the equivalent linear problem of each iteration.

Inner product.- In analogy with the method presented for buckling (see also ref. 10), after each iteration for an estimate $u_{(k)}$ of vibration mode displacements, an eigenvalue estimate $\omega_{(k)}{}^2$ is obtained from the Rayleigh quotient in the form

$$\omega_{(k)}{}^2 = (u_{(k)},\ u_{(k-1)})/(u_{(k)},\ u_{(k)}) \tag{113}$$

where $(u,\tilde{u})$ denotes the inner product of any two kinematically admissible displacement fields.* The inner product represents the work of the inertial loads associated with the displacement field u (or $\tilde{u}$) acting through the displacement field $\tilde{u}$ (or u). It may be written in a form similar to equation (107). In this case, however, there are no inertial free thermal strains and the fourth component of equivalent inertial ring forces FL_4 [eq. (79)] is nonzero. Therefore in place of equation (107), one has†

$$(u,\tilde{u}) = \int_s (F_1\tilde{\xi} + F_2\tilde{\eta} + F_3\tilde{v} + F_4\tilde{\chi})rds + \sum_r (FL_1\tilde{u}_x + FL_2\tilde{u}_y + FL_3u_\phi + FL_4w_\phi) \tag{114}$$

*The inner product is also used to orthogonalize eigenmode estimates with respect to known lower modes in order to force convergence to a higher mode.

†Equation (107) differs in sign (which is immaterial) from the definition given in reference 10.

where the equivalent forces are given by equations (78) and (79) with $L_1^{\bullet}$, $N_x^{\bullet}$, and $N_y^{\bullet}$ replaced by their symmetric components nL_1, nN_x, and nN_y, respectively, and effective loads given by equations (108) and (110) with ω^2 set to unity.

CONCLUDING REMARKS

The governing equations and their method of solution have been presented for stress, buckling, and vibration response of branched, stiffened shells of revolution under axisymmetric and asymmetric loads. In general, the numerical solution is reduced to the solution of a sequence of linear boundary value problems in ordinary differential equations. These are solved by the Zarghamee technique, in which initial conditions for complementary and particular solutions (obtained by forward integration) are chosen so as to satisfy identically the boundary conditions as the corresponding boundaries are passed. This method is more efficient, requiring only half as many complementary solutions for open branch problems, than the more common method of superposition of complementary and particular solutions with arbitrary initial conditions. However, it does not eliminate the problem of "long subintervals" associated with rapid growth of complementary and particular solutions.

A further improvement in the method, which eliminates the long subinterval problem, as well as providing increased efficiency, is currently being evaluated. This new method has been termed the "field method," and it eliminates the calculation of complementary and particular solutions altogether. The efficiency of the field method may make feasible within the scope of the present techniques more advanced problems, such as nonlinear response under unsymmetrical loads.

APPENDIX A

SHELL STIFFNESS COEFFICIENTS

The coefficients λ_{ij}, μ_{ij} in the shell constitutive equations (7) are given by

$$
\begin{aligned}
\lambda_{11} &= -C_{12}^{(0)}\lambda_{13} - C_{12}^{(1)}\lambda_{14} + C_2^{(0)} \\
\lambda_{12} = \lambda_{21} &= -C_{12}^{(1)}\lambda_{13} - C_{12}^{(2)}\lambda_{14} + C_2^{(1)} \\
\lambda_{13} = -\lambda_{31} &= C_{12}^{(0)}\lambda_{33} + C_{12}^{(1)}\lambda_{34} \\
\lambda_{14} = -\lambda_{41} &= C_{12}^{(0)}\lambda_{34} + C_{12}^{(1)}\lambda_{44} \\
\lambda_{22} &= -C_{12}^{(1)}\lambda_{23} - C_{12}^{(2)}\lambda_{24} + C_2^{(2)} \\
\lambda_{23} = -\lambda_{32} &= C_{12}^{(1)}\lambda_{33} + C_{12}^{(2)}\lambda_{34} \\
\lambda_{24} = -\lambda_{42} &= C_{12}^{(1)}\lambda_{34} + C_{12}^{(2)}\lambda_{44} \\
\lambda_{33} &= C_1^{(2)}/\Delta \\
\lambda_{34} = \lambda_{43} &= -C_1^{(1)}/\Delta \\
\lambda_{44} &= C_1^{(0)}/\Delta \\
\mu_{11} &= 4[G^{(0)}G^{(2)} - G^{(1)2}]\mu_{22} \\
\mu_{12} = -\mu_{21} &= 2[G^{(1)} + 2G^{(2)}/R_2]\mu_{22} \\
\mu_{22} &= [G^{(0)} + 4G^{(1)}/R_2 + 4G^{(2)}/R_2{}^2]^{-1}
\end{aligned}
\tag{A-1}
$$

where

$$\Delta = C_1^{(0)}C_1^{(2)} - C_1^{(1)2} \tag{A-2}$$

and for $m = 0, 1,$ or 2

$$
\begin{aligned}
C_1^{(m)} &= C_s^{(m)} + C_{st}^{(m)} \\
C_s^{(m)} &= \int [E_1/(1 - \nu_1\nu_2)] z^m dz \\
C_2^{(m)} &= \int [E_2/(1 - \nu_1\nu_2)] z^m dz \\
C_{12}^{(m)} &= \int [\nu_1 E_2/(1 - \nu_1\nu_2)] z^m dz \\
G^{(m)} &= G_s^{(m)} + G_{st}^{(m)} \\
G_s^{(m)} &= \int E_{12} z^m dz
\end{aligned}
\tag{A-3}
$$

In equations (A-3), the integrals are through the shell wall thickness, and $C_{st}^{(m)}$, $G_{st}^{(m)}$ are stringer contributions given by

$$
\begin{aligned}
C_{st}^{(0)} &= NEA/2\pi r \\
C_{st}^{(1)} &= e_z C_{st}^{(0)} \\
C_{st}^{(2)} &= NEI/2\pi r + e_z^{\,2} C_{st}^{(0)} \\
G_{st}^{(0)} &= G_{st}^{(1)} = 0 \\
G_{st}^{(2)} &= NGJ/8\pi r
\end{aligned}
\tag{A-4}
$$

where e_z is the normal eccentricity of the centroid of stringer cross sections relative to the shell reference surface.

APPENDIX B

SHELLS WITH DOME CLOSURES

Dome closures are treated by deleting a small spherical cap containing the pole and generating appropriate boundary conditions for the artificial edge so created.* These boundary conditions, which represent the deleted cap to first order in the edge radius, are derived in this appendix.

As shown in the main body of this report, all of the problems treated reduce to the solution of a standard linear statics problem with pseudo monoharmonic loads. Furthermore, as noted previously (p. 30), the solution for antisymmetric load components may be obtained from that for equivalent symmetric load problems. Therefore, dome boundary conditions need only be written for the case of linear static response under symmetric pure harmonic loading.

Based on the finiteness of strains, one can derive, in a manner similar to that of reference 21, the following results valid at a pole

$$
\left.
\begin{aligned}
n^2\xi &= 0 \\
\eta + nv &= 0 \\
n\eta + v &= 0 \\
(n^2-1)\chi &= 0 \\
\dot{\xi} &= \chi
\end{aligned}
\right\} \qquad \text{(B-1)}
$$

$$e_1 = \dot{\eta} \qquad \text{(B-2a)}$$

$$e_2 = \dot{\eta} + n\dot{v} \qquad \text{(B-2b)}$$

$$-e_{12} = \pm n\dot{\eta} \qquad \text{(B-2c)}$$

$$\kappa_1 = \pm\dot{\chi} \qquad \text{(B-2d)}$$

$$-\psi = \pm\chi \qquad \text{(B-2e)}$$

$$\ddot{\xi} = \dot{\chi} \pm \dot{\eta}/R_2 \qquad \text{(B-2f)}$$

---

*In evaluation of integrals over the shell meridian, a first-order correction is made to account for the contribution over the deleted cap (see p. 29 of ref. 13).

For the terms with ambiguous signs in equations (B-2), the lower signs apply at a terminal pole, at which $r' = -1$, and the upper signs apply at all other poles, at which $r' = 1$.* The dots above symbols denote differentiation with respect to the radius r. Equations (B-1) and (B-2) are basic equations from which the dome boundary conditions are derived.

Zero'th Harmonic (n = 0)

From equations (B-1) and (B-2), the following relations hold to first order in r at the artificial edge

$$\begin{aligned} \eta &= r\dot{\eta} = re_1 \\ \chi &= r\dot{\chi} = \pm r\kappa_1 \end{aligned} \qquad \text{(B-3)}$$

Using equations (7c,d) and (23c-e), equations (B-3) become

$$\begin{aligned} \pm r\lambda_{33}Q + r\lambda_{34}M_1 - (1 + \lambda_{13})\eta \mp \lambda_{23}\chi &= -r[\lambda_{33}\Theta_1{}^{(0)} + \lambda_{34}\Theta_1{}^{(1)}] \\ \pm r\lambda_{34}Q + r\lambda_{44}M_1 - \lambda_{14}\eta \mp (1 + \lambda_{24})\chi &= -r[\lambda_{34}\Theta_1{}^{(0)} + \lambda_{44}\Theta_1{}^{(1)}] \end{aligned} \qquad \text{(B-4)}$$

Two additional conditions may be derived from the first and third of the equilibrium equations (22a), which for n = 0 reduce to

$$\begin{aligned} (rP)' &= r[r'X_3 - (r/R_2)X_1] \\ (r^2S)' &= r[(r/R_2)L_1 - rX_2] \end{aligned} \qquad \text{(B-5)}$$

Since $(\ )' = (\dot{\ })r'$, in the vicinity of the pole equations (B-5) may be written as

$$\begin{aligned} \dot{\overline{rP}} &= rX_3 + O(r^2) \\ \dot{\overline{r^2S}} &= \pm r^2(L_1/R_2 - X_2) + O(r^3) \end{aligned} \qquad \text{(B-6)}$$

where the loads X_3 and $L_1/R_2 - X_2$ may be taken as their values at the artificial edge. Integration of equations (B-6) between the limits r = 0 and r = r, and neglecting $O(r^2)$, gives at the edge

---

*This is in accordance with the specified description of the reference meridian (see pp. 18-19).

$$
\left.
\begin{aligned}
P &= (r/2)X_3 \\[2ex]
S &= \pm(r/3)(L_1/R_2 - X_2)
\end{aligned}
\right.
\qquad \text{(B-7)}
$$

Equations (B-4) and (B-7) constitute the dome closure boundary conditions for $n = 0$. They are equivalent to equation (25) with

$$
[B] = \pm \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm r\lambda_{33} & 0 & r\lambda_{34} \\ 0 & 0 & 1 & 0 \\ 0 & \pm r\lambda_{34} & 0 & r\lambda_{44} \end{bmatrix} \qquad \text{(B-8a)}
$$

$$
[D] = - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & (1 + \lambda_{13}) & 0 & \pm\lambda_{23} \\ 0 & 0 & 0 & 0 \\ 0 & \lambda_{14} & 0 & \pm(1 + \lambda_{24}) \end{bmatrix} \qquad \text{(B-8b)}
$$

$$
\{L\} = -r \begin{Bmatrix} \mp X_3/2 \\ \lambda_{33}\Theta_I{}^{(0)} + \lambda_{34}\Theta_1{}^{(1)} \\ \pm(X_2 - L_1/R_2)/3 \\ \lambda_{34}\Theta_1{}^{(0)} + \lambda_{44}\Theta_1{}^{(1)} \end{Bmatrix} \qquad \text{(B-8c)}
$$

First Harmonic ($n = 1$)

In this case, equations (B-1) yield at the pole $\xi - r\chi = \dot{\overline{\xi - r\chi}} = \eta + v = 0$, so that one may write at a small distance from the pole

$$
\xi - r\chi = O(r^2) \qquad \text{(B-9a)}
$$

$$
\eta + v = r(\dot{\eta} + \dot{v}) + O(r^2) \qquad \text{(B-9b)}
$$

Equation (B-9a) yields immediately the first-order condition

$$
\xi - r\chi = 0 \qquad \text{(B-10)}
$$

However, equation (B-9b) does not yield any new information, since substitution of equation (B-2b) for $\dot{\eta} + \dot{v}$, and elimination of e_2 through equation (23d) results in the identity $\eta + v = \eta + v$. In place of equation (B-9b), equations (B-2a) and (B-2c) can be used to form the relation

$$re_1 \pm re_{12} = O(r^2) \qquad \text{(B-11)}$$

Using equations (7c,f) and (23c-f), and neglecting $O(r^2)$, equation (B-11) becomes

$$\pm(\lambda_{23} - \mu_{12})(\xi/r - \chi) - (\lambda_{13} + \lambda_{32}/R_2)(\eta + v) \pm r\lambda_{33}Q \pm r\mu_{22}S + r\lambda_{34}M_1$$
$$= -r[\lambda_{33}\Theta_1^{(0)} + \lambda_{34}\Theta_1^{(1)}] \qquad \text{(B-12)}$$

At this point it is desirable to evaluate the term $\xi/r - \chi$. For this purpose it is necessary to compute $\ddot{\overline{\xi - r\chi}}$ at the pole. Differentiation gives

$$\ddot{\overline{\xi - r\chi}} = \ddot{\xi} - r\ddot{\chi} - 2\dot{\chi} \qquad \text{(B-13)}$$

At the pole $r = 0$, so that substitution of equation (B-2f) gives

$$\ddot{\overline{\xi - r\chi}} = -\dot{\chi} \pm \dot{\eta}/R_2 \qquad \text{(B-14)}$$

Since, as already noted, $\xi - r\chi = \dot{\overline{\xi - r\chi}} = 0$ at the pole, one may write at a small distance from the pole

$$\xi - r\chi = (r^2/2)(-\dot{\chi} \pm \dot{\eta}/R_2) + O(r^3) \qquad \text{(B-15)}$$

Therefore, to first order in r, the following relation holds

$$\xi/r - \chi = r(-\dot{\chi} \pm \dot{\eta}/R_2)/2 = -r(e_{12}/R_2 \pm \kappa_1)/2 \qquad \text{(B-16)}$$

in which equations (B-2c,d) have been used. Substituting equations (7d,f) and (23c-f) into equation (B-16) and solving the resulting equation for $\xi/r - \chi$ gives

$$\xi/r - \chi = \{\pm(\lambda_{14} + \lambda_{24}/R_2)(\eta + v)/2 - (r/2)[\lambda_{34}Q + \mu_{22}S/R_2$$
$$\pm \lambda_{44}M_1 \pm \lambda_{34}\Theta_1^{(0)} \pm \lambda_{44}\Theta_1^{(1)}]\}/[1 + (\lambda_{24} - \mu_{12}/R_2)/2] \qquad \text{(B-17)}$$

Substitution of equation (B-17) into equation (B-12) to eliminate $\xi/r - \chi$ gives the desired boundary condition

$$\pm r(A_1\lambda_{33} - A_2\lambda_{34})Q \pm r\mu_{22}(A_1 - A_2/R_2)S + r(A_1\lambda_{34} - A_2\lambda_{44})M_1$$
$$+ [A_2(\lambda_{14} + \lambda_{24}/R_2) - A_1(\lambda_{13} + \lambda_{23}/R_2)](\eta + v)$$
$$= r[(A_2\lambda_{34} - A_1\lambda_{33})\Theta_1{}^{(0)} + (A_2\lambda_{44} - A_1\lambda_{34})\Theta_1{}^{(1)}] \tag{B-18}$$

where
$$\begin{aligned} A_1 &= 1 + (\lambda_{24} - \mu_{12}/R_2) \\ A_2 &= (\lambda_{23} - \mu_{12})/2 \end{aligned} \tag{B-19}$$

Two additional conditions may be derived from the equilibrium equations (22a). For n = 1, the second and third equations and the first and fourth equations may be combined to give

$$\begin{aligned} r'[\dot{\overline{r(Q - S)}}] &= r[X_2 - r'X_1 - (r/R_2)X_3] \\ r'[\dot{\overline{r(rP + M_1)}}] &= r\{r[r'X_3 - (r/R_2)X_1] - r'L_1 - L_2\} \\ &\quad + (r/R_2)[r(Q - S)] \end{aligned} \tag{B-20}$$

In the vicinity of the pole, equations (B-20) may be written as

$$\begin{aligned} [\dot{\overline{r(Q - S)}}] &= r(\pm X_2 - X_1) + O(r^2) \\ [\dot{\overline{r(rP + M_1)}}] &= -r(L_1 \pm L_2) + O(r^2) \end{aligned} \tag{B-21}$$

where the loads X_1, X_2, L_1, L_2 may be taken as their values at the artificial edge. Integration of equations (B-21) between the limits r = 0 and r = r, and neglecting $O(r^2)$ gives at the edge

$$\begin{aligned} Q - S &= (r/2)(\pm X_2 - X_1) \\ rP + M_1 &= -(r/2)(L_1 \pm L_2) \end{aligned} \tag{B-22}$$

Equations (B-10), (B-18), and (B-22) constitute the dome closure boundary conditions for n = 1. They are equivalent to equation (25) with

$$[B] = \pm\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \pm r(A_1\lambda_{33}-A_2\lambda_{34}) & \pm r\mu_{22}(A_1-A_2/R_2) & r(A_1\lambda_{34}-A_2\lambda_{44}) \\ 0 & 1 & -1 & 0 \\ r & 0 & 0 & 1 \end{bmatrix} \tag{B-23a}$$

$$[D] = \begin{bmatrix} 1 & 0 & 0 & -r \\ 0 & A_3 & A_3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{B-23b}$$

$$\{L\} = r\begin{Bmatrix} 0 \\ (A_2\lambda_{34}-A_1\lambda_{33})\Theta_1{}^{(0)} + (A_2\lambda_{44}-A_1\lambda_{34})\Theta_1{}^{(1)} \\ (\pm X_2-X_1)/2 \\ -(L_1\pm L_2)/2 \end{Bmatrix} \tag{B-23c}$$

where

$$A_3 = A_2(\lambda_{14} + \lambda_{24}/R_2) - A_1(\lambda_{13} + \lambda_{23}/R_2) \tag{B-24}$$

Higher Harmonics ($n \geq 2$)

In this case, equations (B-1) yield at the pole, $\xi = \eta = v = \chi = \dot{\xi} = 0$. In view of equations (B-2), the following relations therefore hold to first order in r at the artificial edge

$$\begin{aligned} \xi &= 0 \\ \eta &= r\dot{\eta} = re_1 \\ v &= r\dot{v} = r(ne_2 \pm e_{12})/n^2 \\ \chi &= r\dot{\chi} = \pm r\kappa_1 \end{aligned} \tag{B-25}$$

Using equations (7c,d,f) and (23c-f), equations (B-25) can be put in the form of equation (25), where

$$[B] = \pm \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \pm r\lambda_{33} & 0 & r\lambda_{34} \\ 0 & 0 & r\mu_{22} & 0 \\ 0 & \pm r\lambda_{34} & 0 & r\lambda_{44} \end{bmatrix} \tag{B-26a}$$

$$[D] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1+\lambda_{13}+n^2\lambda_{23}/2R_2) & -n(\lambda_{13}+\lambda_{23}/R_2) & \mp(1-n^2/2)\lambda_{23} \\ 0 & \pm n(1-\mu_{12}/2R_2) & 0 & n\mu_{12}/2 \\ 0 & -(\lambda_{14}+n^2\lambda_{24}/2R_2) & -n(\lambda_{14}+\lambda_{24}/R_2) & \mp[1+(1-n^2/2)\lambda_{24}] \end{bmatrix} \tag{B-26b}$$

$$\{L\} = -r \begin{Bmatrix} 0 \\ \lambda_{33}\Theta_1{}^{(0)} + \lambda_{34}\Theta_1{}^{(1)} \\ 0 \\ \lambda_{34}\Theta_1{}^{(0)} + \lambda_{44}\Theta_1{}^{(1)} \end{Bmatrix} \tag{B-26c}$$

APPENDIX C

SUPPORTING LEMMAS FOR ZARGHAMEE METHOD

In this appendix several basic aspects of the Zarghamee procedure are clarified in order to provide some formal justification for this method.

Supplemental Initial Conditions

As noted in the discussion of the Zarghamee method (p. 24), the supplemental initial conditions chosen should insure linear independence of the complementary solutions. This question is clarified by the following lemma: Any supplemental condition of the form

$$[\alpha][U_0] + [\beta][W_0] = [I] \tag{C-1}$$

where $[\alpha]$ and $[\beta]$ are 4x4 matrices, insures linear independence of the four complementary solutions on open branches.

The proof is as follows: As is well known, linear independence of a set of complementary solutions of a system of ordinary linear differential equations implies linear independence everywhere.* Therefore it is sufficient to show linear independence at the initial point alone. Linear dependence of the complementary solutions at the initial point is equivalent to the existence of a non-null 4x1 constant matrix $\{c\}$ such that

$$\begin{aligned} [U_0]\{c\} &= \{0\} \\ [W_0]\{c\} &= \{0\} \end{aligned} \tag{C-2}$$

Postmultiplication of equation (C-1) by $\{c\}$ shows that equations (C-2) imply $\{c\} = \{0\}$, contradicting the requirement of non-null $\{c\}$. Q.E.D.†

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*Since a linear combination of solutions of a system of homogeneous linear equations is also a solution, if it vanishes at one point, by the uniqueness theorem of initial-value problems, it vanishes everywhere.

†This argument may also be used to prove linear independence of the additional four complementary solutions on a closed branch. It does not, however, prove linear independence of all eight complementary solutions on a closed branch.

Nonsingularity of [W] and [Z]

The Zarghamee method involves the inversion of [W] at the final points of subintervals on open branches [see eqs.(46) and (49a)], and inversion of [W] and [Z] at the final points of subintervals on a closed branch [see eqs. (53) and (55)]. In this section it is shown that these final point matrices are always nonsingular, so that their inverses exist.

Let us denote the final value of [W] for a typical subinterval by $[W_1]$. Consider the columns of the 8x4 matrix of complementary solutions $[{}^{U}_{W}]$ as the static solutions for a shell whose extent is the subinterval considered and which has null surface loads and initial boundary conditions given by equation (47a) and the final boundary conditions

$$[W] = [W_1] \tag{C-3}$$

Since equations (C-3) are displacement conditions, it follows from the uniqueness theorem of linear elasticity that the corresponding solutions are unique. Now assume that $[W_1]$ is singular, i.e. one of its columns, say the i-th column, is a linear combination of the other three.* Since equations (47a) and the differential equations are linear and homogeneous, and the solutions are unique, it follows that the i-th solution must be the same linear combination of the other three solutions. Thus the assumption of singular $[W_1]$ contradicts the result of the preceding section, in which it is shown that the supplemental initial conditions insure the linear independence of complementary solutions.

The above argument can be applied without change as well to [Z], for which equation (47a) is replaced by equation (54). [For the first subinterval on a closed branch, equations (47a) and (54) are replaced by equations (58b).]

Kinematic Constraints on a Closed Branch

For the Zarghamee method to be used on a closed branch, it is necessary that all eight complementary solutions be linearly independent. In this section, the conditions under which eight linearly independent complementary solutions exist are derived.

As noted on pages 23 and 27, the 8x8 matrix of complementary solution vectors $\underset{\sim}{Y}_c^{(k)}$ (k = 1,···,8) on a closed branch is partitioned into four

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*The possibility that one of its columns is null is excluded since this would imply that the corresponding complementary solution is identically zero.

4x4 submatrices,*

$$[\underset{\sim}{Y}_c^{(k)}] = \left[\begin{array}{c|c} U & V \\ \hline W & Z \end{array}\right] \tag{C-4}$$

A necessary and sufficient condition for the $\underset{\sim}{Y}_c^{(k)}$ to be linearly independent is that the determinant of the matrix $[\underset{\sim}{Y}_c^{(k)}]$ be nonzero. As noted previously (p. 59), the value of this determinant at any point will suffice, and its value Δ at the initial point of a generic subinterval is calculated below. The first subinterval of a closed branch may be excluded from further discussion, since in this case it is obvious from equations (58a,b) and (C-4) that $\Delta = 1$.

For any other subinterval of a closed branch, Δ is evaluated as follows. Assuming that initially U is nonsingular, the first row of the right-hand side of equation (C-4) premultiplied by WU^{-1} is subtracted from the second row to give†

$$\Delta = \begin{vmatrix} U & V \\ W & Z \end{vmatrix} = \begin{vmatrix} U & V \\ 0 & Z-WU^{-1}V \end{vmatrix}$$

$$= |U| \cdot |Z-WU^{-1}V| \tag{C-5}$$

In order to evaluate the second factor of this result, note that from equation (47a),

$$WU^{-1} = -\tilde{D}^{-1}B \tag{C-6}$$

Substituting equation (C-6) and using equation (54) to eliminate BV, one obtains

$$Z - WU^{-1}V = Z - \tilde{D}^{-1}\hat{D}Z$$

$$= \tilde{D}^{-1}(\tilde{D} - \hat{D})Z \tag{C-7}$$

---

*For the sake of simplicity, the brackets and braces used for 4x4 and 4x1 matrices, respectively, are omitted in the remainder of this section.

†Henceforth in this discussion, it is assumed that U,V,W,Z are evaluated at the initial point of the subinterval.

Evaluating the difference $\tilde{D} - \hat{D}$ from equations (48a), (49a), and (55), equation (C-7) becomes

$$Z - WU^{-1}V = \tilde{D}^{-1}B(V_1Z_1^{-1} - U_1W_1^{-1})Z \tag{C-8}$$

where, here the subscript 1 denotes values at the end of the preceding closed branch subinterval. Substitution of equation (C-8) into equation (C-5) gives

$$\Delta = |U|\cdot|Z|\cdot|B|\cdot|V_1Z_1^{-1} - U_1W_1^{-1}|/|\tilde{D}| \tag{C-9}$$

Since from equation (C-6), $|W| = -|B|\cdot|U|/|\tilde{D}|$, an alternate form of equation (C-9) is

$$\Delta = -|W|\cdot|Z|\cdot|V_1Z_1^{-1} - U_1W_1^{-1}| \tag{C-10}$$

It follows from equation (C-9) that if B is singular, $\Delta = 0$ and therefore linearly independent complementary solutions do not exist in the subinterval. Since equation (C-9) is derived without the use of supplemental initial conditions, this conclusion is true regardless of the choice of these conditions. Since singular B is equivalent to kinematic constraint, for the Zarghamee method kinematic constraints are not allowed on a closed branch except at the closure point.

Equation (C-10) is used to guide the choice of supplemental initial conditions on a closed branch. The simplest possible choice consistent with the requirement that initially W and Z should be nonsingular is used, viz. $W = Z = I$ [cf. eq. (56a)].

APPENDIX D

CALCULATION OF SHELL STRESSES

After solution of the shell equations for the eight basic force and displacement variables [fig. 2(a)], SRA 100 (and SRA 200) proceeds to calculate all components of the three-dimensional stress tensor except the transverse normal stress σ_z, which is negligible for a thin shell. The equations used for the calculation of these stresses are presented in this appendix.

According to the Love-Kirchhoff thin shell hypothesis, neglecting terms of order z/R relative to unity,* the primary (i.e., in-surface) shell stresses for an orthotropic wall are given by

$$\begin{aligned}
\sigma_s &= [E_1/(1 - \nu_1\nu_2)][\varepsilon_1 + \nu_2\varepsilon_2 + z(\kappa_1 + \nu_2\kappa_2) - (\theta_1 + \nu_2\theta_2)] \\
\sigma_\phi &= [E_2/(1 - \nu_1\nu_2)][\varepsilon_2 + \nu_1\varepsilon_1 + z(\kappa_2 + \nu_1\kappa_1) - (\theta_2 + \nu_1\theta_1)] \qquad \text{(D-1)} \\
\sigma_{s\phi} &= E_{12}(\varepsilon_{12} + 2z\kappa_{12})
\end{aligned}$$

In order to compute these stress components from eqs. (D-1), all that is needed are the stretching $(\varepsilon_1,\varepsilon_2,\varepsilon_{12})$ and bending $(\kappa_1,\kappa_2,\kappa_{12})$ strains of the reference surface. After solution of the differential equations for the basic force (y) and displacement (z) vectors (see pp. 22-30), these strains are given by eqs. (23d-f) and (7c,d,f) with the aid of eqs. (23c) and (8).

The transverse shear and normal stresses may then be obtained by integration through the shell thickness of the three-dimensional equations of equilibrium. In terms of symmetrical stress amplitudes (table I) these equations are, assuming no body forces and neglecting terms of order z/R relative to unity,

$$\partial\sigma_{sz}/\partial z = -\partial\sigma_s/\partial s - (n/r)\sigma_{s\phi} + (r'/r)(\sigma_\phi - \sigma_s) \qquad \text{(D-2a)}$$

$$\partial\sigma_{\phi z}/\partial z = -\partial\sigma_{s\phi}/\partial s + (n/r)\sigma_\phi - 2(r'/r)\sigma_{s\phi} \qquad \text{(D-2b)}$$

$$\partial\sigma_z/\partial z = -(1/r)\partial(r\sigma_{sz})/\partial s - (n/r)\sigma_{\phi z} + \sigma_s/R_1 + \sigma_\phi/R_2 \qquad \text{(D-2c)}$$

where R_1 is the meridional radius of curvature. Inspection of eqs. (D-2) shows that for thin shells the transverse stresses $\sigma_{sz},\sigma_{\phi z},\sigma_z$ can be

*Here, R represents either principal radius of curvature (R_1 or R_2) of the reference surface.

significant (i.e., >> z/R times a primary stress) only in a boundary-layer zone, where s-differentiation is tantamount to multiplication by a factor much larger than 1/R, or if n/r >> 1/R. On the other hand, even in a boundary-layer zone, for the thin shell hypothesis to remain valid it is necessary that $\sigma_{sz},\sigma_{\phi z},\sigma_z << \sigma_s,\sigma_\phi,\sigma_{s\phi}$. Accepting this as being the case, it is clear from eqs. (D-2) that the transverse normal stress σ_z is negligible, even in boundary-layer zones and hence need not be computed.

In practice, eqs. (D-2a,b) are integrated with respect to z starting at the shell wall inner face. At this face σ_{sz} and $\sigma_{\phi z}$ are given as $-X_1$ and $-X_2$ by eqs. (3a,b) for an attached elastic foundation or are simply zero if no foundation exists. In evaluating the s-derivatives on the right-hand sides of eqs. (D-2a,b) the differential equations (22) are used to obtain derivatives of response variables, whereas derivatives of wall properties and thermal loads are neglected.

APPENDIX E

GENERAL BUCKLING EQUATIONS

In this appendix the treatment of the buckling equations (based on moderate rotation theory) without restriction of structural geometry or loading is presented. This development may therefore be taken as the basis for both the bifurcation analysis of linearized asymmetric equilibrium states (SRA 101) and the bifurcation analysis of nonlinear axisymmetric equilibrium states (SRA 201). Although set in a more general context, this development follows along the same lines as that of reference 11.

Eigenvalue Equations

Buckling equations without restriction on structural geometry or loading distribution have been presented in reference 18. As presented there, the variational form of these equations for the buckling mode variables u, σ, ε in terms of the prebuckling equilibrium state variables $u_0(\lambda)$, $\sigma_0(\lambda)$ at a load factor λ (for a proportional loading) is

$$\begin{aligned} \varepsilon &= L_1(u) + L_{11}(u_0,u) \\ \sigma &= H(\varepsilon) \\ \sigma\cdot\delta\varepsilon_0 &+ \sigma_0\cdot L_{11}(u,\delta u) - \lambda q_1(u)\cdot\delta u = 0 \end{aligned} \tag{E-1}$$

where

$$\delta\varepsilon_0 \equiv L_1(\delta u) + L_{11}(u_0,\delta u) \tag{E-2}$$

In order to apply the iteration method for bifurcation problems, it is necessary to search for eigenvalues in a sufficiently small neighborhood of a load estimate $\lambda = \lambda_0$, so that in this neighborhood the prebuckling state variables $u_0(\lambda)$, $\sigma_0(\lambda)$ have a linear dependence on λ. Setting $\lambda = \lambda_0 + \mu$, one has, to first order in μ,†

$$\begin{aligned} u_0(\lambda) &= u_0 + \mu u_0^{(1)} \\ \sigma_0(\lambda) &= \sigma_0 + \mu\sigma_0^{(1)} \end{aligned} \tag{E-3}$$

Substitution of equations (E-3) into equations (E-1) and (E-2) yields the

†Henceforth, variables with the subscript 0 are assumed to be evaluated at $\lambda = \lambda_0$.

linear eigenvalue problem with eigenvalues μ_i, $i = 1,2,\ldots$(arranged in the order of increasing absolute value)

$$\varepsilon = L_1(u) + L_{11}(u_0,u) + \mu L_{11}(u_0{}^{(1)},u) \tag{E-4a}$$

$$\sigma = H(\varepsilon) \tag{E-4b}$$

$$\begin{aligned}&\sigma\cdot\delta\varepsilon_0 + \sigma_0\cdot L_{11}(u,\delta u) - \lambda_0 q_1(u)\cdot\delta u\\&\qquad + \mu[\sigma\cdot L_{11}(u_0{}^{(1)},\delta u) + \sigma_0{}^{(1)}\cdot L_{11}(u,\delta u) - q_1(u)\cdot\delta u] = 0\end{aligned} \tag{E-4c}$$

Orthogonality of eigenfunctions is obtained by setting $\delta u = u_j$ in equations (E-4c) evaluated for the i-th eigenvalue, interchanging i and j, and subtracting the resulting two equations. With the aid of equations (E-2), (E-4a) and (E-4b), this gives the result

$$(\mu_i - \mu_j)(u_i,\sigma_i;u_j,\sigma_j) = 0 \tag{E-5}$$

where the inner (scalar) product is defined by*

$$\begin{aligned}(u_i,\sigma_i;u_j,\sigma_j) \equiv\ &\sigma_0{}^{(1)}\cdot L_{11}(u_i,u_j) + \sigma_i\cdot L_{11}(u_0{}^{(1)},u_j)\\&+ \sigma_j\cdot L_{11}(u_0{}^{(1)},u_i) - q_1(u_i)\cdot u_j\end{aligned} \tag{E-6}$$

It is noted that the second and third terms on the right-hand side of equation (E-6) represent prebuckling deformations (i.e., rotations), and the last term represents conservative live loads (i.e., a normal pressure field). Also, the inner product is seen to be a symmetric functional, i.e., $(u_i,\sigma_i;u_j,\sigma_j) = (u_j,\sigma_j;u_i,\sigma_i)$. From equation (E-5) it follows that eigenfunctions corresponding to distinct eigenvalues μ_i,μ_j are orthogonal, i.e.

$$(u_i,\sigma_i;u_j,\sigma_j) = 0 \quad , \quad i \neq j \tag{E-7}$$

Similarly, evaluation of equation (E-4c) for the i-th eigenvalue with $\delta u = u_i$ gives the expression for the eigenvalues in terms of the eigenfunctions, viz.

$$\mu_i = -[\sigma_0\cdot L_2(u_i) + \sigma_i\cdot\varepsilon_i - \lambda_0 q_1(u_i)\cdot u_i]/[\sigma_0{}^{(1)}\cdot L_2(u_i) - q_1(u_i)\cdot u_i] \tag{E-8}$$

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*For $\lambda_0 = \lambda_c$, the critical load, this definition of the inner product is equal to the functional $F^{(1)}(u_i,u_j)$ defined in reference 18.

The right-hand side of equation (E-8) (with the index i deleted) defines the Rayleigh quotient, which may be used to obtain an estimate μ for μ_1 if estimates u, σ, ε for $u_1, \sigma_1, \varepsilon_1$ are available. However, in applying equation (E-8) in this way, care must be taken that u, ε and μ satisfy the kinematic equation (E-4a), in addition to σ, ε satisfying equation (E-4b). Such kinematically admissible functions admit the eigenfunction expansions

$$u = \sum_{i=1}^{\infty} A_i u_i$$
$$\sigma = \sum_{i=1}^{\infty} A_i \sigma_i \tag{E-9}$$

Forming from equations (E-9) the inner product $(u,\sigma;u_j,\sigma_j)$ shows that, in view of equation (E-7),

$$A_i = (u,\sigma;u_i,\sigma_i)/(u_i,\sigma_i;u_i,\sigma_i) \tag{E-10}$$

Iterative Solution of Eigenvalue Equations

The iteration method consists of successive solution of the following modified form of equations (E-4)

$$\varepsilon_{(k)} = L_1(u_{(k)}) + L_{11}(u_0,u_{(k)}) + L_{11}(u_0^{(1)},u_{(k-1)}) \tag{E-11a}$$

$$\sigma_{(k)} = H(\varepsilon_{(k)}) \tag{E-11b}$$

$$\sigma_{(k)}\cdot\delta\varepsilon_0 + \sigma_0\cdot L_{11}(u_{(k)},\delta u) - \lambda_0 q_1(u_{(k)})\cdot\delta u + \sigma_{(k-1)}\cdot L_{11}(u_0^{(1)},\delta u)$$
$$+ \sigma_0^{(1)}\cdot L_{11}(u_{(k-1)},\delta u) - q_1(u_{(k-1)})\cdot\delta u = 0 \tag{E-11c}$$

where the subscript (k) denotes the solution after k iterations. The estimates $u_{(k)}$, $\sigma_{(k)}$, $\varepsilon_{(k)}$ cannot be used directly in the Rayleigh quotient to obtain an estimate $\mu_{(k)}$, nor do they admit eigenfunction expansions, since in general $u_{(k)}$, $\varepsilon_{(k)}$, and $\mu_{(k)}$ would not satisfy the kinematic equations (E-4a). It is therefore necessary to relate $\sigma_{(k)}$, $\varepsilon_{(k)}$ to modified variables $\hat{\sigma}_{(k)}$, $\hat{\varepsilon}_{(k)}$ such that $u_{(k)}$, $\hat{\sigma}_{(k)}$, $\hat{\varepsilon}_{(k)}$ and $\mu_{(k)}$ are kinematically admissible, i.e.

$$\hat{\varepsilon}_{(k)} = L_1(u_{(k)}) + L_{11}(u_0,u_{(k)}) + \mu_{(k)} L_{11}(u_0^{(1)},u_{(k)}) \tag{E-12a}$$

$$\hat{\sigma}_{(k)} = H(\hat{\varepsilon}_{(k)}) \tag{E-12b}$$

Taking the difference of equations (E-12a,b) and (E-11a,b) gives

$$\begin{aligned} \hat{\sigma}_{(k)} - \sigma_{(k)} &= H(\hat{\varepsilon}_{(k)} - \varepsilon_{(k)}) \\ \hat{\varepsilon}_{(k)} - \varepsilon_{(k)} &= L_{11}(u_0^{(1)}, \mu_{(k)} u_{(k)} - u_{(k-1)}) \end{aligned} \tag{E-13}$$

from which it is observed that the difference $\hat{\sigma}_{(k)} - \sigma_{(k)}$ is of the order of prebuckling rotations, the square of which is negligible for our moderate rotation equations. Forming the work term $\hat{\sigma}_{(k)} \cdot \hat{\varepsilon}_{(k)}$ (required in the Rayleigh quotient) from equations (E-13), and neglecting terms of the order of the square of prebuckling rotations, gives

$$\hat{\sigma}_{(k)} \cdot \hat{\varepsilon}_{(k)} = \sigma_{(k)} \cdot \varepsilon_{(k)} + 2\sigma_{(k)} \cdot L_{11}(u_0^{(1)}, \mu_{(k)} u_{(k)} - u_{(k-1)}) \tag{E-14}$$

Setting $\delta u = u_i$ in equation (E-11c), $\delta u = u_{(k)}$ in equation (E-4c) evaluated for the i-th eigenvalue, and subtracting the results gives the relation

$$(u_i, \sigma_i; u_{(k)}, \sigma_{(k)}) = (u_i, \sigma_i; u_{(k-1)}, \sigma_{(k-1)})/\mu_i \tag{E-15}$$

From equation (E-6) it follows that

$$\begin{aligned} &(u_i, \sigma_i; u_{(k)}, \hat{\sigma}_{(k)}) - (u_i, \sigma_i; u_{(k)}, \sigma_{(k)}) \\ &\quad = (\hat{\sigma}_{(k)} - \sigma_{(k)}) \cdot L_{11}(u_0^{(1)}, u_i) \end{aligned} \tag{E-16}$$

Since as observed previously $\hat{\sigma}_{(k)} - \sigma_{(k)}$ is of the order of prebuckling rotations, this difference of inner products is of the order of the square of prebuckling rotations and therefore negligible. Hence, it follows from equations (E-10) and (E-15) that the expansion coefficients A_{ik} for the functions $u_{(k)}$, $\hat{\sigma}_{(k)}$ satisfy the relation

$$A_{ik} = A_{i,k-1}/\mu_i \tag{E-17}$$

from which it follows that as $k \to \infty$

$$\begin{aligned} u_{(k)} &\to u_1 \\ \hat{\sigma}_{(k)} &\to \sigma_1 \end{aligned} \tag{E-18}$$

proving convergence of the iteration method.

Substituting $u_{(k)}$, $\hat{\sigma}_{(k)}$, $\hat{\varepsilon}_{(k)}$ for u_i, σ_i, ε_i in the Rayleigh quotient given in equation (E-8), eliminating $\hat{\sigma}_{(k)} \cdot \hat{\varepsilon}_{(k)}$ through use of equation (E-14), eliminating $\sigma_{(k)} \cdot \varepsilon_{(k)}$ through use of equation (E-11c) with $\delta u = u_{(k)}$, and solving for the corresponding eigenvalue estimate $\mu_{(k)}$ gives the result

$$\mu_{(k)} = (u_{(k)}, \sigma_{(k)}; u_{(k-1)}, \sigma_{(k-1)})/(u_{(k)}, \sigma_{(k)}; u_{(k)}, \sigma_{(k)}) \qquad \text{(E-19)}$$

Since, as already noted, the inner product is insensitive to the difference $\sigma_{(k)} - \hat{\sigma}_{(k)}$, to the accuracy of our moderate rotation equations, $\mu_{(k)}$ is also given by equation (E-19) with $\sigma_{(k)}$ replaced by $\hat{\sigma}_{(k)}$. To this form can be applied the completeness relation

$$(u_{(k)}, \hat{\sigma}_{(k)}; u_{(k-1)}, \hat{\sigma}_{(k-1)}) = \sum_{i=1}^{\infty} (u_i, \sigma_i; u_i, \sigma_i) A_{ik} A_{i,k-1} \qquad \text{(E-20)}$$

giving in view of equation (E-17) the result

$$\mu_{(k)} = \sum_{i=1}^{\infty} \mu_i (u_i, \sigma_i; u_i, \sigma_i) A_{ik}{}^2 / \sum_{i=1}^{\infty} (u_i, \sigma_i; u_i, \sigma_i) A_{ik}{}^2 \qquad \text{(E-21)}$$

Equation (E-21) shows that, by use of the Rayleigh quotient, $\mu_{(k)}$ converges to μ_1 at a much faster rate than $u_{(k)}$, $\hat{\sigma}_{(k)}$ converge to u_1, σ_1.

Finally, since $\mu_{(k)} \to \mu_1$, and from equation (E-17) $\mu_1 u_{(k)} \to u_{(k-1)}$, it follows from equations (E-13) that $\sigma_{(k)} \to \hat{\sigma}_{(k)}$. Hence, in spite of its inadmissibility

$$\sigma_{(k)} \to \sigma_1 \qquad \text{(E-22)}$$

Thus, in the practical application of the iteration procedure, at no time is it necessary to work with the modified variables $\hat{\sigma}_{(k)}$ or $\hat{\varepsilon}_{(k)}$.

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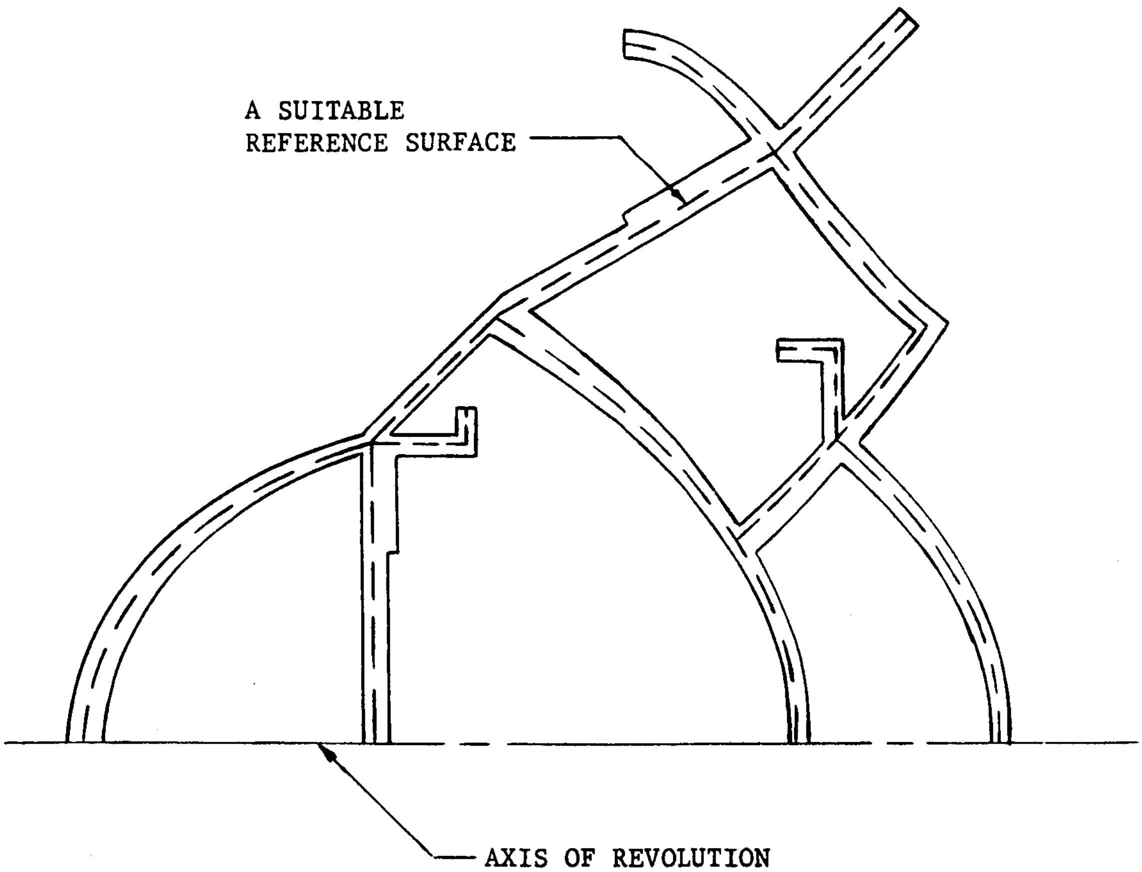
TABLE I

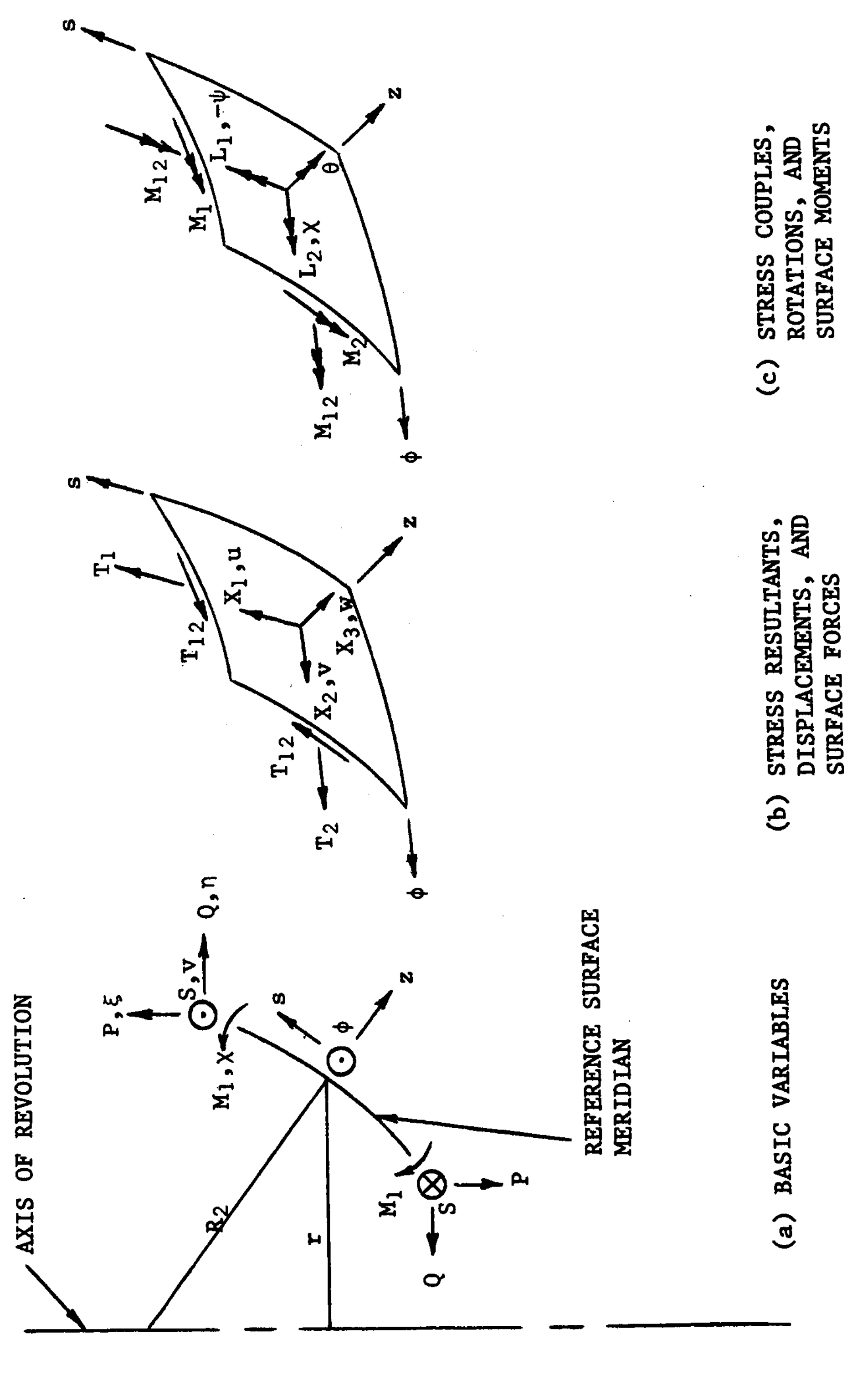
CIRCUMFERENTIAL VARIATION OF HARMONIC VARIABLES

	Symmetric		Antisymmetric	
	cos $n\phi$	sin $n\phi$	cos $n\phi$	sin $n\phi$
Shell Load Variables	X_1, X_3 L_2 $\Theta_1^{(0)}, \Theta_2^{(0)}$ $\Theta_1^{(1)}, \Theta_2^{(1)}$ θ_1, θ_2	X_2 L_1 $\Theta_{12}^{(0)}$ $\Theta_{12}^{(1)}$	X_2 L_1 $\Theta_{12}^{(0)}$ $\Theta_{12}^{(1)}$	X_1, X_3 L_2 $\Theta_1^{(0)}, \Theta_2^{(0)}$ $\Theta_1^{(1)}, \Theta_2^{(1)}$ θ_1, θ_2
Ring Load Variables	F_x, F_y N_ϕ $EA\theta_\phi$	F_ϕ N_x, N_y	F_ϕ N_x, N_y	F_x, F_y N_ϕ $EA\theta_\phi$
Shell Response Variables	P, Q T_1, T_2 M_1, M_2 u, w ξ, η χ $\varepsilon_1, \varepsilon_2$ κ_1, κ_2 $\sigma_s, \sigma_\phi, \sigma_z, \sigma_{sz}$	S T_{12} M_{12} v ψ, θ ε_{12} κ_{12} $\sigma_{s\phi}, \sigma_{\phi z}$	S T_{12} M_{12} v ψ, θ ε_{12} κ_{12} $\sigma_{s\phi}, \sigma_{\phi z}$	P, Q T_1, T_2 M_1, M_2 u, w ξ, η χ $\varepsilon_1, \varepsilon_2$ κ_1, κ_2 $\sigma_s, \sigma_\phi, \sigma_z, \sigma_{sz}$
Ring Response Variables	T_ϕ M_x, M_y u_x, u_y w_ϕ ε_ϕ κ_x, κ_y	M_ϕ u_ϕ w_x, w_y τ	M_ϕ u_ϕ w_x, w_y τ	T_ϕ M_x, M_y u_x, u_y w_ϕ ε_ϕ κ_x, κ_y



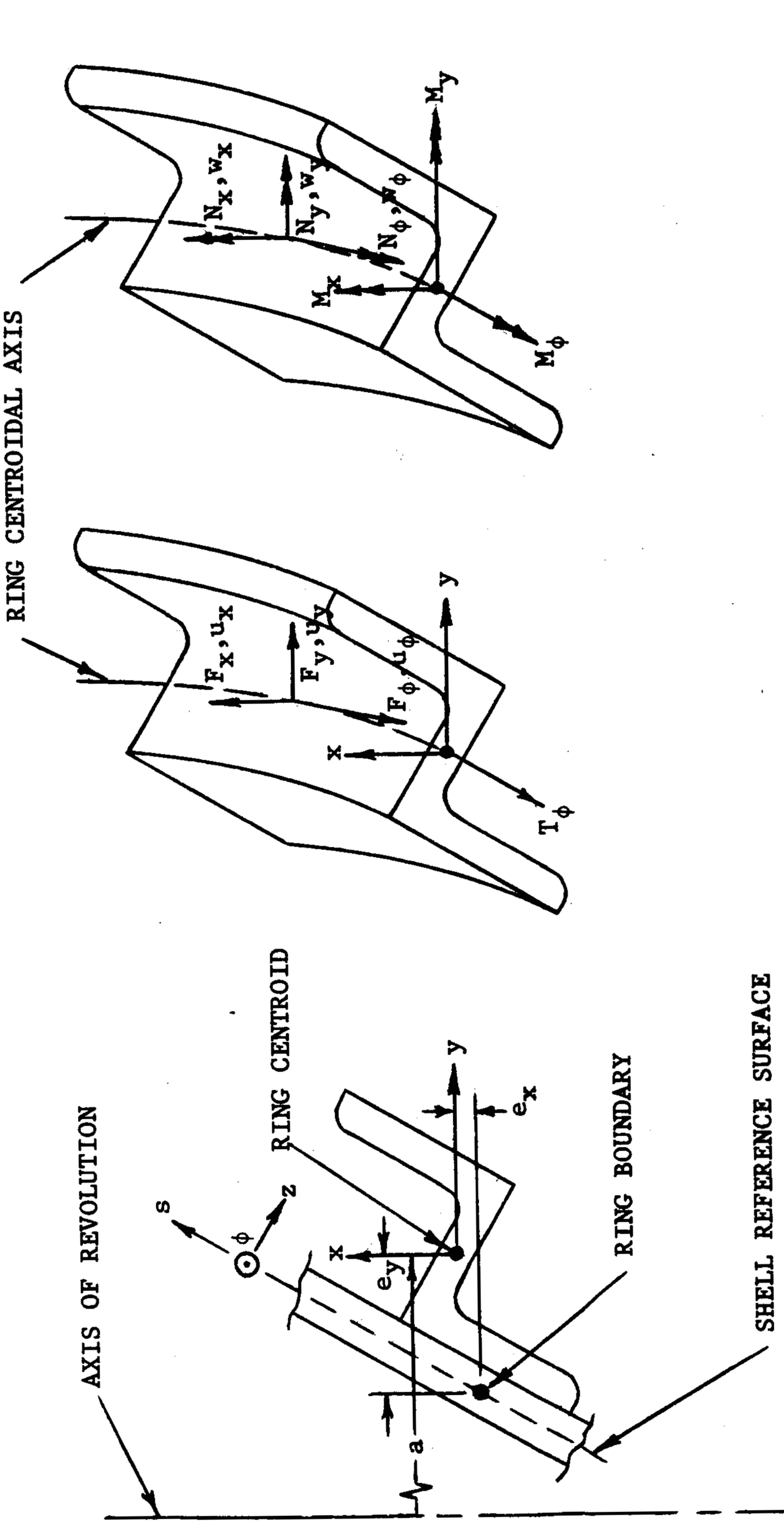
NOTE: The reference surface may be chosen as any convenient continuous surface within or near the shell wall.

FIGURE 1. HYPOTHETICAL BRANCHED SHELL PROFILE (WITH FIVE BRANCH POINTS AND ONE CLOSED BRANCH)



NOTE: Coordinates s, ϕ, z form a right-handed system. The radius of curvature R_2 is reckoned positive only if the local positive z-direction points away from the axis of revolution.

FIGURE 2. POSITIVE DIRECTIONS FOR SHELL DISPLACEMENTS, FORCES, AND LOADS

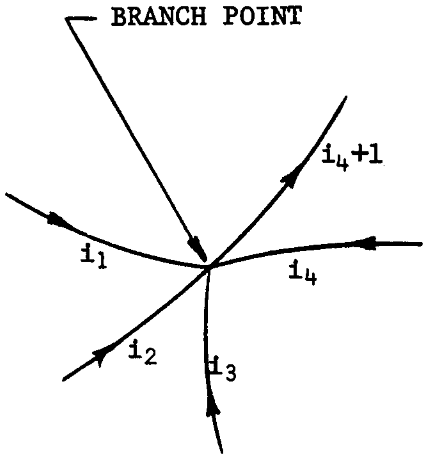


(a) RING x,y-AXES, ECCENTRICITIES e_x,e_y AND RADIUS a

(b) HOOP FORCE T_ϕ, DISPLACEMENTS u_x,u_y,u_ϕ AND LINE LOADS (FORCE) F_x,F_y,F_ϕ

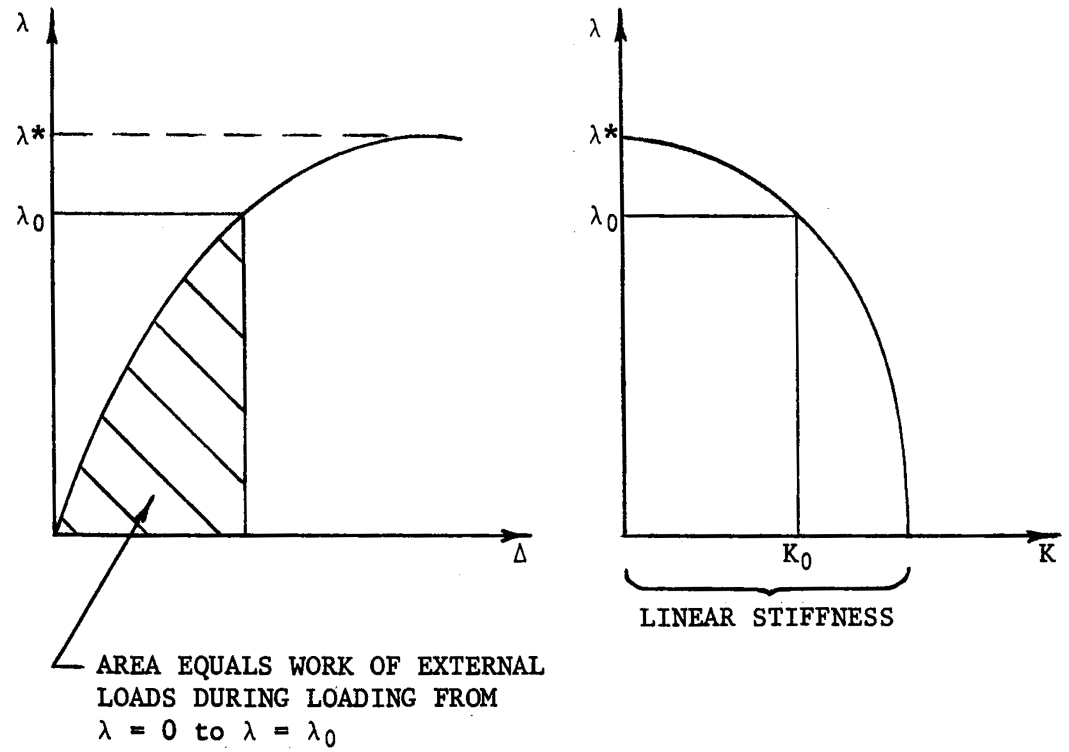
(c) BENDING MOMENTS M_x,M_y, TORSIONAL MOMENT M_ϕ, ROTATIONS w_x,w_y,w_ϕ AND LINE LOADS (MOMENT) N_x,N_y,N_ϕ

FIGURE 3. POSITIVE DIRECTIONS FOR RING VARIABLES



NOTE: Arrows indicate direction of increasing s. Subintervals i_1, i_2, i_3, and i_4 (here $J = 4$) enter the branch point, whereas subinterval i_4+1 exits the branch point. The branch point is denoted as $i_j,1$ on entering subintervals and $i_J+1,0$ on the exiting subinterval.

FIGURE 4. HYPOTHETICAL BRANCH POINT BOUNDARY (WITH FOUR ENTERING SUBINTERVALS)



(a) LOAD VS. WORK DEFLECTION

(b) LOAD VS. STIFFNESS
$K = d\lambda/d\Delta$

FIGURE 5. TYPICAL PREBUCKLING LOADING DIAGRAMS